# Introduction to quantum computing

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&
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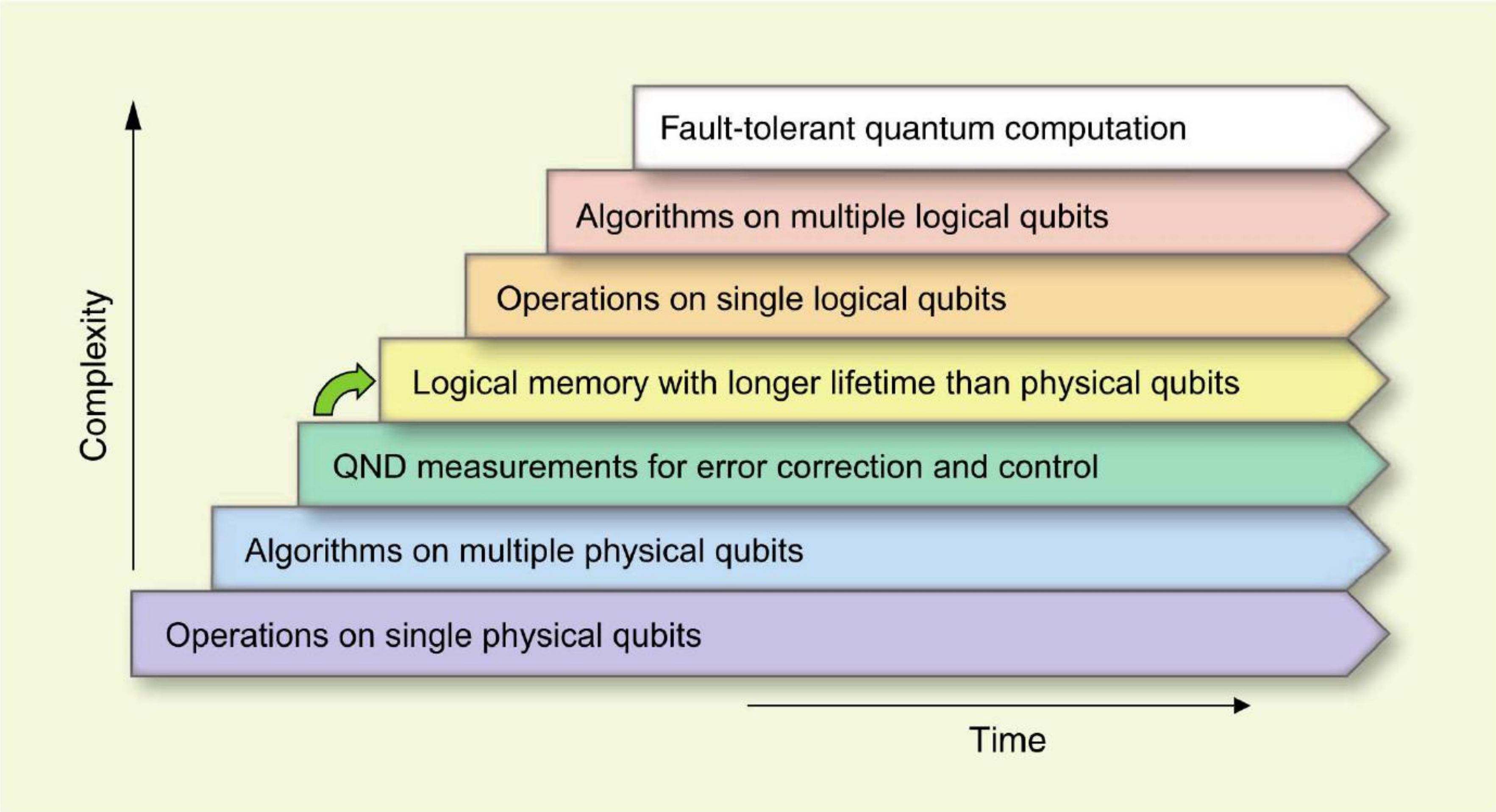
# QUANTUM INFORMATION PROCESSING

## What is next?

- Interesting quantum devices in the next 10 years:
  = complexity that CANNOT EVER be classically simulated
  (> 50 qubits or equivalent)

- Outstanding questions:
  what level of quantum error correction(QEC) needed?
  how much overhead QEC?
  what's the best architecture?
  what are the useful and achievable (on short term) applications?

# ROAD-MAP TOWARDS FAULT-TOLERANT QUANTUM COMPUTATION




M.H. Devoret & R.J. Schoelkopf, Science 339, 1169-1174 (2013).

# OUTLINE

- ❑ Classical vs quantum error correction
- ❑ Theory of quantum error correction
- ❑ Stabilizer formalism
- ❑ Fault-tolerant QEC
- ❑ Fault-tolerant logical gates
- ❑ Concatenation and threshold theorem
- ❑ A brief introduction to surface codes
- ❑ A brief introduction to continuous variable codes

# MATERIAL

- « Quantum computation and quantum information »
  M.A. Nielsen & I.L. Chuang
- Lecture notes by John Preskill (Caltech)
  http://www.theory.caltech.edu/people/preskill/ph229/
- Surface codes: Toward practical large-scale quantum computation
  A.G. Fowler et al., PRA 86,032324 (2012)
- Quantum error correction for quantum memories
  B.M. Terhal, Rev. Mod. Phys. 87, 307 (2015).
- PhD Thesis, Joachim Cohen, ENS Paris (Feb 2017).

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# QUANTUM ERROR CORRECTION

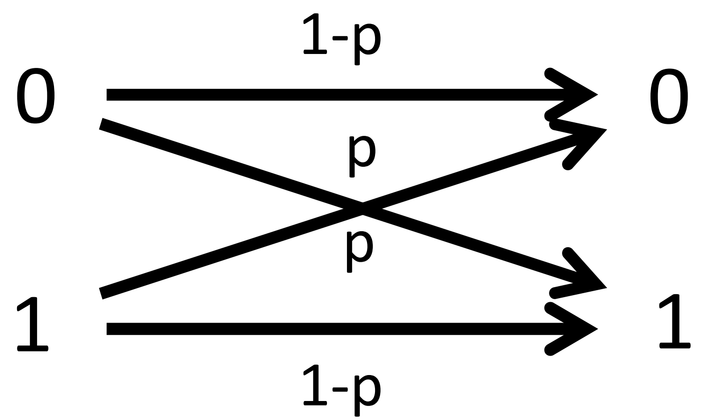
**Scheme for reducing decoherence in quantum computer memory**

Peter W. Shor*
*AT&T Bell Laboratories, Room 2D-149, 600 Mountain Avenue, Murray Hill, New Jersey 07974*
(Received 17 May 1995)

- Decoherence: not a fondamental objection to quantum computation;
- Model continuous decoherence as discrete error channels;
- Redundantly encode quantum information in an entangled state of a multi-qubit system and perform quantum error correction.

# CLASSICAL NOISE, CLASSICAL ERROR CORRECTION

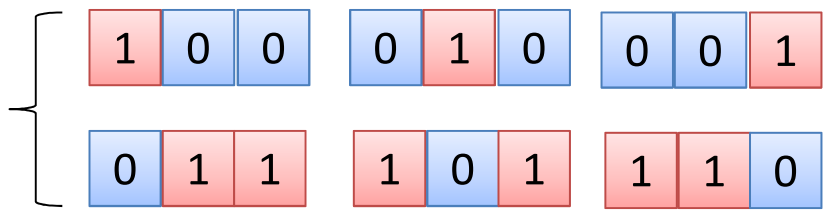
**Classical noise: bit-flip errors**




Basics of **classical** error correction: redundancy


« repetition code »

1-bit errors tractable by **majority vote**:



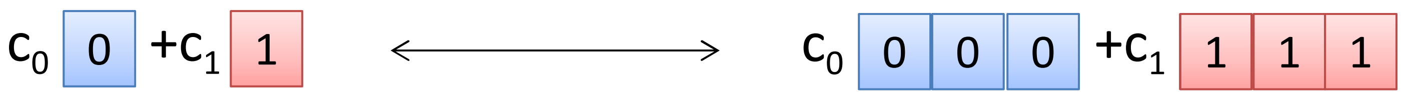

Probability of incorrectible 2-bit errors: $< 3p^2$

(p error probability per unit time)

# QUANTUM VS CLASSICAL ERROR CORRECTION

**Objective**: Protect any superposition state $c_0|0> + c_1|1>$ without any knowledge of $c_0$ and $c_1$.

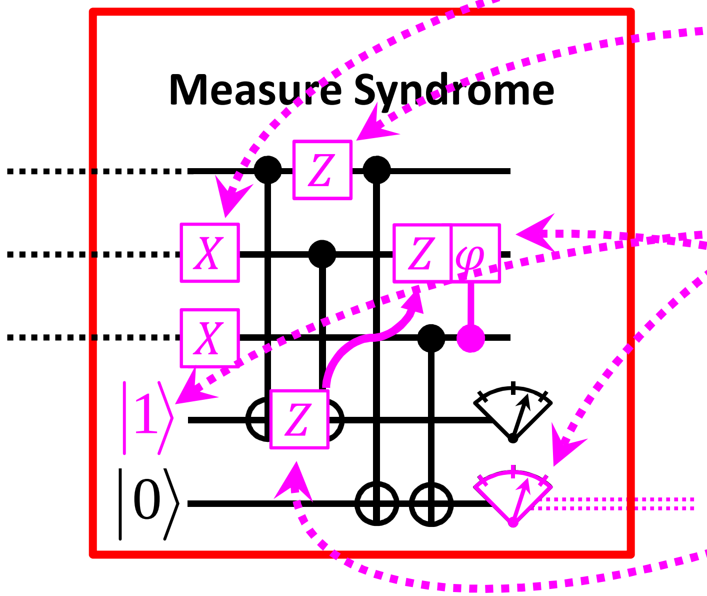
**Quantum** error correction: bit-flip errors




- Majority vote erases the information.
- 1-bit errors tractable by **parity measurement**: $Z_1Z_2$ and $Z_2Z_3$
- Four outcomes: (++) No errors, (-+) error on Q1, (+-) error on Q3, (--) error on Q2.

# THE BIT-FLIP CODE IN PRACTICE

What are the Failure Modes?




**Failures**

1. Double Errors
2. Uncorrectable Errors
3. Readout Errors
4. Ancilla Prep.
5. Undesired Couplings
6. Forward Propagation

$$P_{FAIL} = \sum_{i=1}^{6} P_i F_i$$

Courtesy of R. Schoelkopf

# QEC BEYOND BIT-FLIP ERRORS

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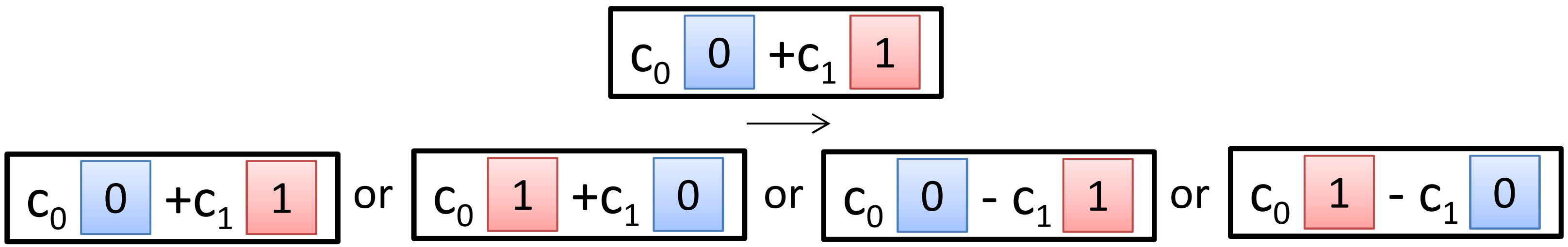
One needs to correct four possible error channels:
I,X,Z,Y=iXZ



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# QEC BEYOND BIT-FLIP ERRORS

## Quantum noise: interaction with environment

**A general error mechanism:**

$$\mathcal{E}\left(\rho_s\right)=\mathrm{tr}_{\mathrm{env}}\left[U_\tau\left(\rho_s\otimes\rho_{\mathrm{env}}\right)U_\tau^\dagger\right]=\sum_k E_k\rho_s E_k^\dagger$$

with $$\sum_k E_k^\dagger E_k = I.$$

## Remark

The choice of the Kraus operators is not unique:

$$\tilde{E}_\mu=\sum_\nu u_{\mu,\nu}E_\nu, \qquad \left(u_{\mu,\nu}\right)\ \text{unitary}$$

satisfies $$\sum_\mu E_\mu\rho E_\mu^\dagger=\sum_\mu \tilde{E}_\mu\rho\tilde{E}_\mu^\dagger \qquad \forall\rho$$

# EXAMPLES

## Pure dephasing

$$\mathcal{E}_{\varphi}\left(\rho\right)=\boldsymbol{E}_0\rho\boldsymbol{E}_0^{\dagger}+\boldsymbol{E}_1\rho\boldsymbol{E}_1^{\dagger},$$

$$\boldsymbol{E}_0=\sqrt{1-p}\boldsymbol{I},\quad \boldsymbol{E}_1=\sqrt{p}\boldsymbol{\sigma}_z,\quad p=\tau/T_{\varphi}$$

## T1 Relaxation

$$\mathcal{E}_{T1}\left(\rho\right)=\boldsymbol{E}_0\rho\boldsymbol{E}_0^{\dagger}+\boldsymbol{E}_1\rho\boldsymbol{E}_1^{\dagger},$$

$$\boldsymbol{E}_0=\left|0\right\rangle\left\langle 0\right|+\sqrt{1-p}\left|1\right\rangle\left\langle 1\right|,\quad \boldsymbol{E}_1=\sqrt{p}\left|0\right\rangle\left\langle 1\right|,\quad p=\tau/T1$$

# QEC BEYOND BIT-FLIP ERRORS

## Theory of QEC

Similarly to an error channel, the error correction (measuement and feedback) can be modeled by a quantum operation:

$$\rho \to \mathcal{R}\left(\rho\right) = \sum_{k} \boldsymbol{R}_{k} \rho \boldsymbol{R}_{k}^{\dagger}$$

This corrects an error channel $\rho \to \mathcal{E}\left(\rho\right)$ if for any $\rho_c$ in the code space

$$\mathbb{R} \circ \mathcal{E}\left(\rho_c\right) = \rho_c .$$

# QUANTUM ERROR CORRECTION CRITERIA

## Theorem:

- Let $\mathcal{C}$ be a quantum code, with a basis $\{|\phi_k\rangle\}$ for the code subspace.
- Suppose $\mathcal{E}$ is an error channel with elements $E_k$.
- A necessary and sufficient condition for the existence of error recovery operations is

$$\langle \phi_k | E_i^\dagger E_j | \phi_l \rangle = \alpha_{ij} \delta_{kl}$$

where $(\alpha_{ij})$ is hermitian.

## Interpretation

Orthogonal codewords remain orthogonal after the errors

$$E_i|\phi_k\rangle \perp E_j|\phi_l\rangle$$

# QEC BEYOND BIT-FLIP ERRORS

**Theorem: discretization of error channels**

If the operation $\mathcal{R}$ corrects the error channel $\mathcal{E}$, it corrects any other error channel $\mathcal{F}$ whose elements $\boldsymbol{F_k}$ are linear combinations of elements $\boldsymbol{E_k}$ with complex coefficients:

$$\mathcal{R}\circ\mathcal{E}\left(\rho\right)=\rho \quad \Rightarrow \quad \mathcal{R}\circ\mathcal{F}\left(\rho\right)=\rho$$

**Corollary: case of qubits**

It sufficies to correct the operations $\left\{\boldsymbol{I},\boldsymbol{\sigma_x},\boldsymbol{\sigma_z},\boldsymbol{\sigma_y}=\boldsymbol{i\sigma_x\sigma_z}\right\}$ to correct for any single-qubit errors.

# FULL QUANTUM ERROR CORRECTION

Four possible error channels for each qubit: I, X, Z, Y=iXZ

$c_0$ 0 $+c_1$ 1

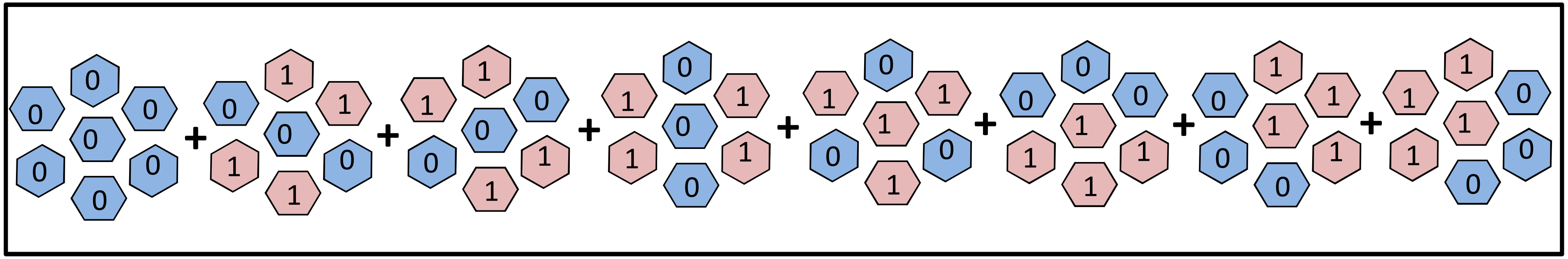
→

$c_0$ 0 $+c_1$ 1 or $c_0$ 1 $+c_1$ 0 or $c_0$ 0 $-\ c_1$ 1 or $c_0$ 1 $-\ c_1$ 0

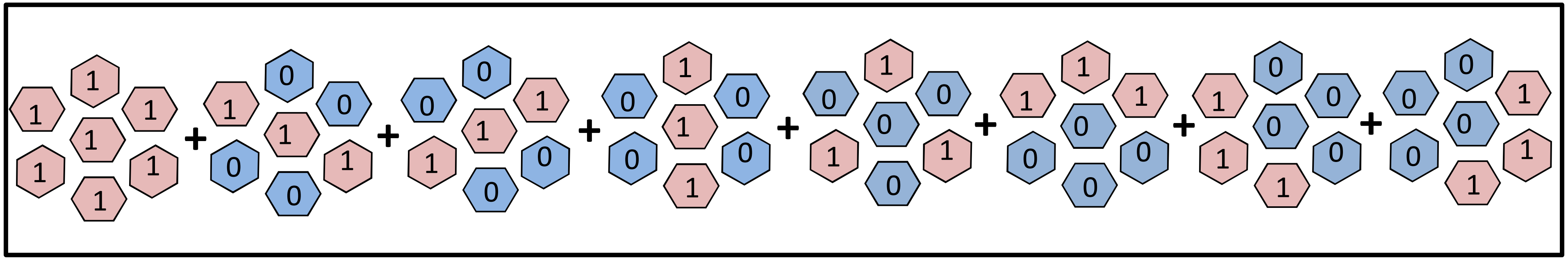
At least five qubits to make all these errors tractable

**7-qubit Steane code:**

0 →



1 →

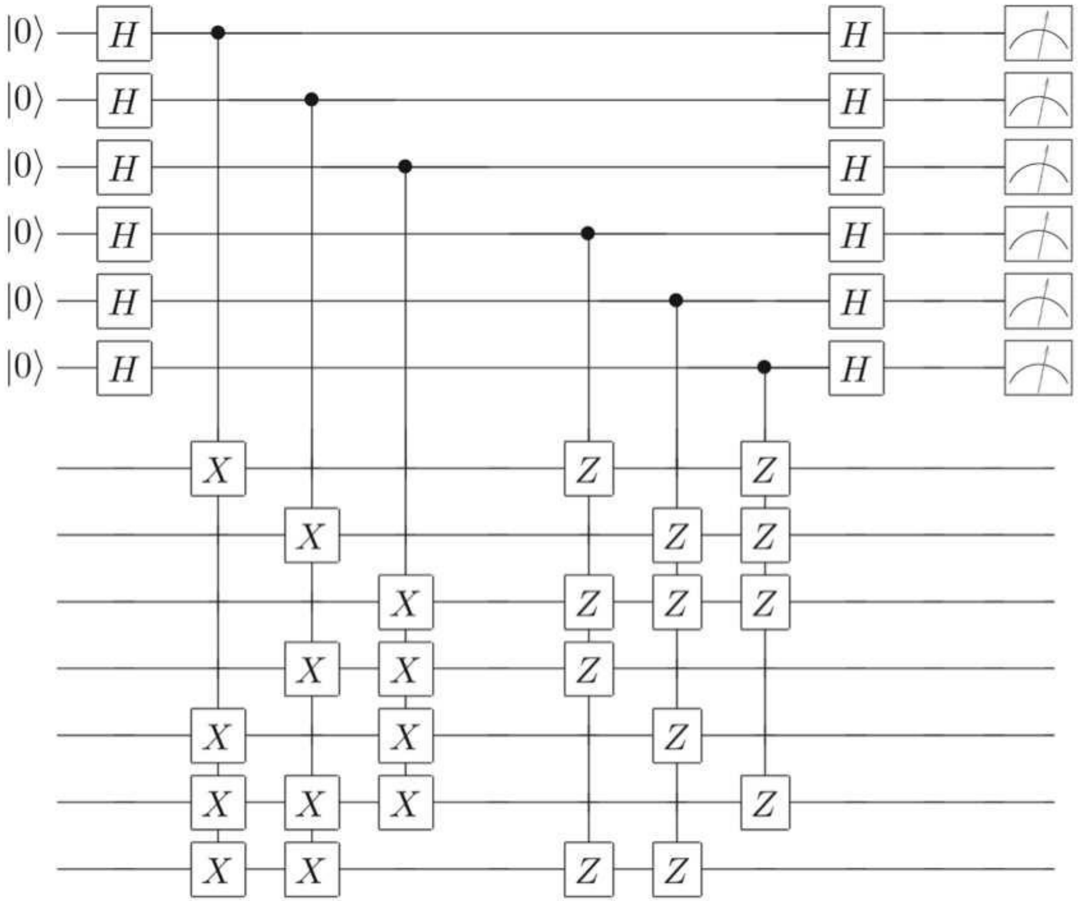


# FULL QUANTUM ERROR CORRECTION

$$|0_L\rangle = \frac{1}{\sqrt{8}}\Big[|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle\Big]$$

$$|1_L\rangle = \frac{1}{\sqrt{8}}\Big[|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle\Big]$$

**Single round of error correction**



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# STABILIZER QUANTUM ERROR CORRECTING CODES

**Idea:** quantum states could be represented by operators that stabilize them, e.g. the EPR state $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is the unique state such that

$$\boldsymbol{X}_1\boldsymbol{X}_2|\psi\rangle = |\psi\rangle, \qquad \boldsymbol{Z}_1\boldsymbol{Z}_2|\psi\rangle = |\psi\rangle$$

**Pauli Group:** $\mathbb{G}_{\boldsymbol{n}} = \{\boldsymbol{I}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}\}^{\otimes \boldsymbol{n}} \otimes \{\pm 1, \pm i\}$

**Properties:** $\boldsymbol{P}^2 = \pm \boldsymbol{I}, \qquad \boldsymbol{PQ} = \pm \boldsymbol{QP}, \qquad \boldsymbol{PP}^\dagger = \boldsymbol{I}.$

**Stabilizer group:** subgroup $\mathbb{S}$ of $\mathbb{G}_{\boldsymbol{n}}$, all elements commute with each other and does not contain $-\boldsymbol{I}$.

**Stabilizer generators:** Minimal set of operators $\boldsymbol{g}_{\boldsymbol{k}}$ that generate $\mathbb{S}$:

$$\mathbb{S} = <\boldsymbol{g}_1, \boldsymbol{g}_2, \cdots, \boldsymbol{g}_{\boldsymbol{r}}> \subseteq \mathbb{G}_{\boldsymbol{n}}$$

**Stabilizer subspace:** subspace of dimension $2^{n-r}$

$$\mathcal{H}_{\mathbb{S}} = \{ |\psi\rangle \,|\, \boldsymbol{S}|\psi\rangle = |\psi\rangle \ \textbf{ for } \boldsymbol{S} \in \mathbb{S} \}$$

# ERROR-CORRECTION CONDITIONS FOR STABILIZER CODES

## Theorem:

Let $\mathbb{S}$ be the stabilizer for a quantum code. Suppose $\{E_k\}$ is a set of operators in $\mathbb{G}_n$. A sufficient condition for the correctability of these errors is that one of the following holds

- $E_a^\dagger E_b \in \mathbb{S}$ ,
- There is an $M \in \mathbb{S}$ that anti-commutes with $E_a^\dagger E_b$.

## Proof:

Case 1: $$\langle \phi_j | E_a^\dagger E_b | \phi_k \rangle = \langle \phi_j | \phi_k \rangle = \delta_{jk}$$

Case 2: $$\langle \phi_j | E_a^\dagger E_b | \phi_k \rangle = \langle \phi_j | E_a^\dagger E_b M | \phi_k \rangle = -\langle \phi_j | M E_a^\dagger E_b | \phi_k \rangle = -\langle \phi_j | E_a^\dagger E_b | \phi_k \rangle$$

and therfore $\langle \phi_j | E_a^\dagger E_b | \phi_k \rangle = 0$

# STABILIZER CODES: EXAMPLES

## Bit-flip code:

Taking $\mathbb{S} = < Z_1Z_2, Z_2Z_3 >$ for error operators $E_k = X_k$

$$Z_1Z_2X_1X_3 = -X_1X_3Z_1Z_2, \quad Z_2Z_3X_1X_3 = -X_1X_3Z_2Z_3$$
$$Z_1Z_2X_2X_3 = -X_2X_3Z_1Z_2, \quad Z_1Z_3X_2X_3 = -X_2X_3Z_1Z_3$$
$$Z_1Z_3X_1X_2 = -X_1X_2Z_1Z_3, \quad Z_2Z_3X_1X_2 = -X_1X_2Z_2Z_3$$

## Steane code:

Stabilizer group:

$$\mathbb{S} = < Z_1Z_3Z_5Z_7, Z_2Z_3Z_6Z_7, Z_4Z_5Z_6Z_7, X_1X_3X_5X_7, X_2X_3X_6X_7, X_4X_5X_6X_7 >$$

Error operators:

$$E_{1,\cdots,7} = X_{1,\cdots,7}, \quad E_{8,\cdots,14} = Y_{1,\cdots,7}, \quad E_{15,\cdots,21} = Z_{1,\cdots,7}$$

# STABILIZER CODES: LOGICAL OPERATIONS

## Definition

Operators in $\mathbb{G}_n$ that commute with $\mathbb{S}$ : they act on the $2^{n-r}$-dimensional stabilizer subspace.

## Steane code:

Stabilizer group:

$$\mathbb{S} = \langle Z_1Z_3Z_5Z_7, Z_2Z_3Z_6Z_7, Z_4Z_5Z_6Z_7, X_1X_3X_5X_7, X_2X_3X_6X_7, X_4X_5X_6X_7 \rangle$$

Logical operators:

$$\bar{Z} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7, \qquad \bar{X} = X_1X_2X_3X_4X_5X_6X_7, \qquad \bar{Y} = i\bar{X}\bar{Z}$$

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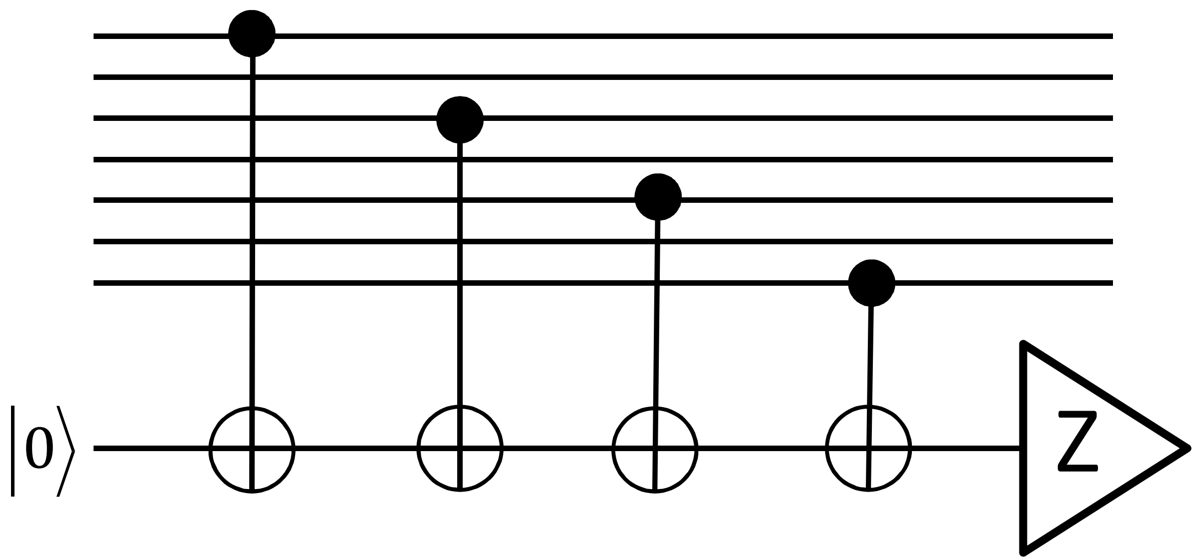
# FAULT-TOLERANCE

**Central idea:** through operations, one should not introduce new error channels not taken into account by QEC. In particular, one should avoid propagation/amplification of errors

Example of parity measurements: simplest circuit to measure the parity $Z_1Z_3Z_5Z_7$ for the Steane code.



# NOT FAULT-TOLERANT

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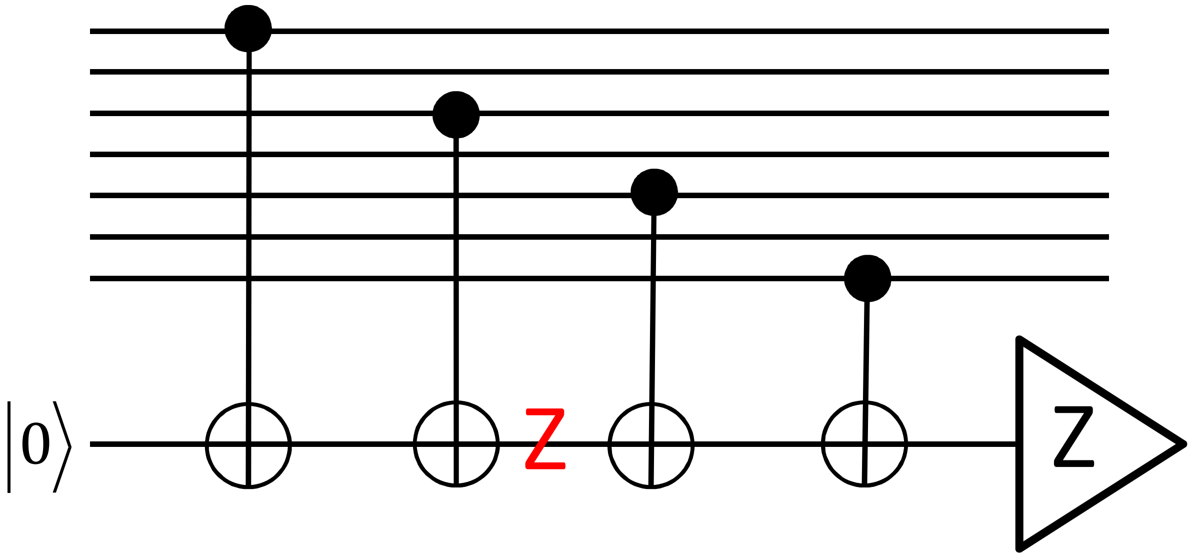


A phase-flip of the ancilla qubit propagates to memory qubits.

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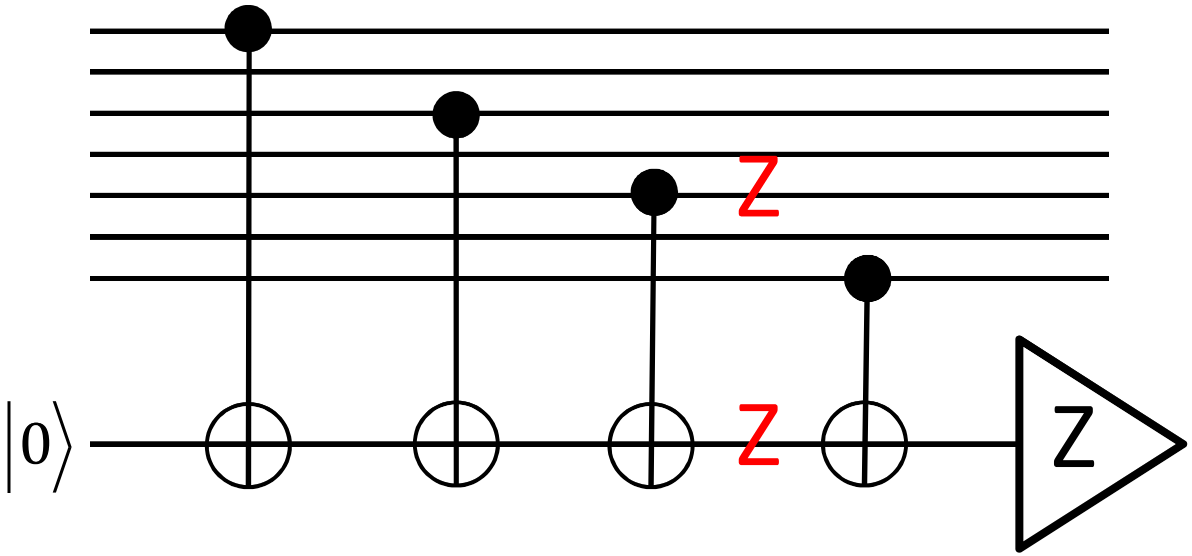


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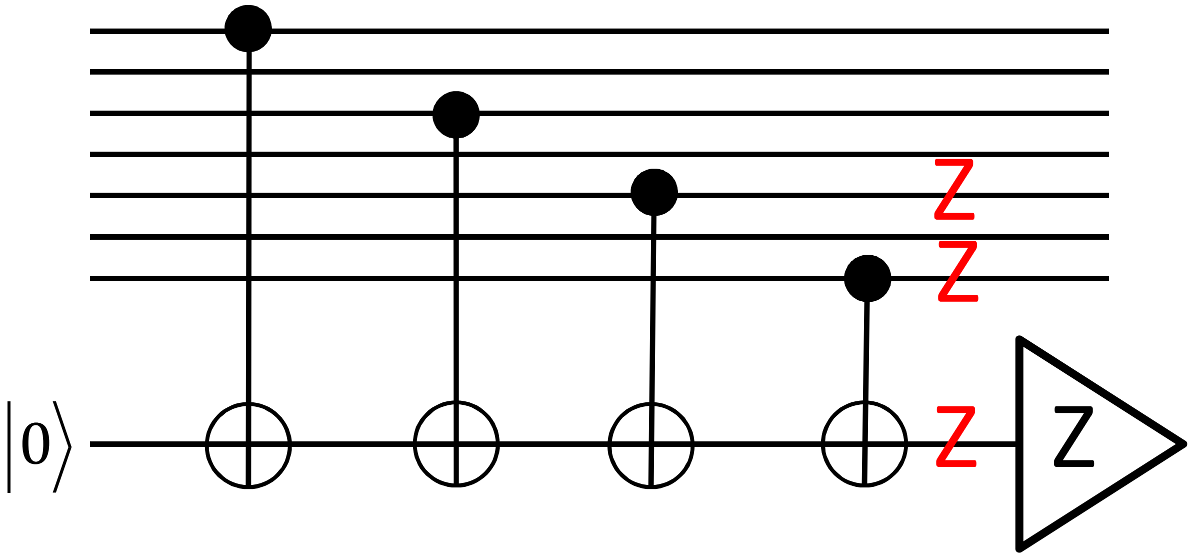
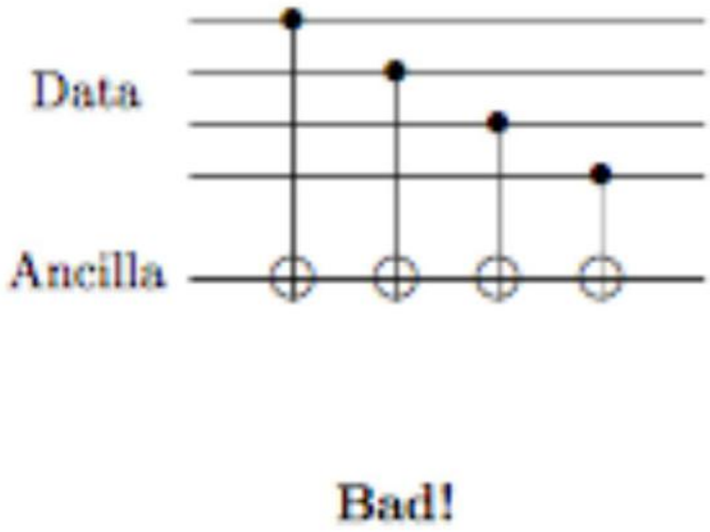
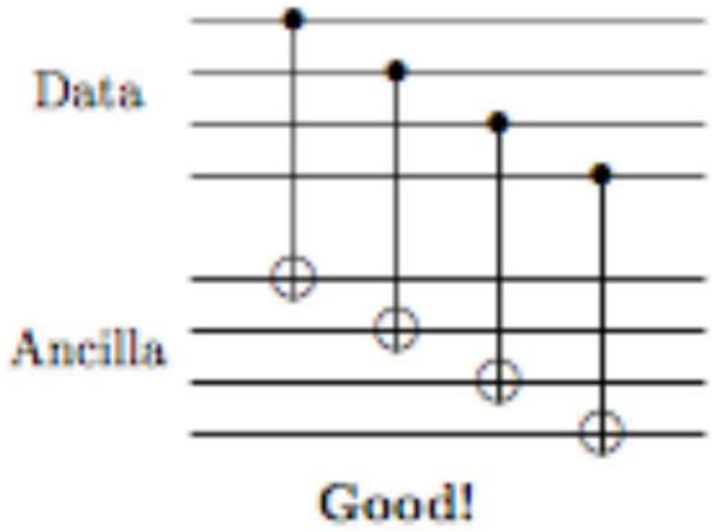



A phase-flip of the ancilla qubit propagates to memory qubits.

# TOWARDS A SOLUTION

Idea N1: transversal operations




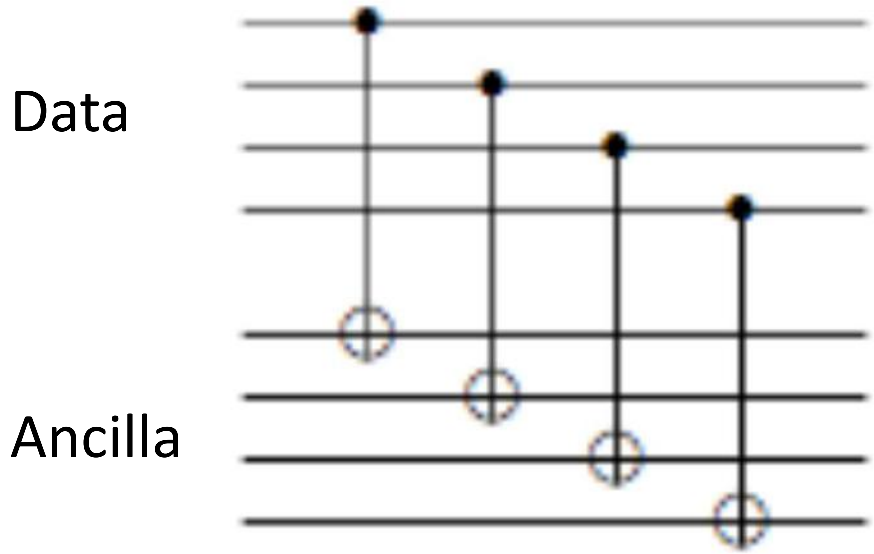

- Each ancilla qubit couples to no more than one memory qubit.
- We readout more than the required information (ancillas get entangled to the codeword).

J. Preskill, Fault-tolerant quantum computation, 1997.

# TOWARDS A SOLUTION

Idea N2: encoding ancillas

$$|\mathrm{Shor}\rangle = \frac{1}{\sqrt{8}} \sum_{\text{even } v} |v\rangle$$

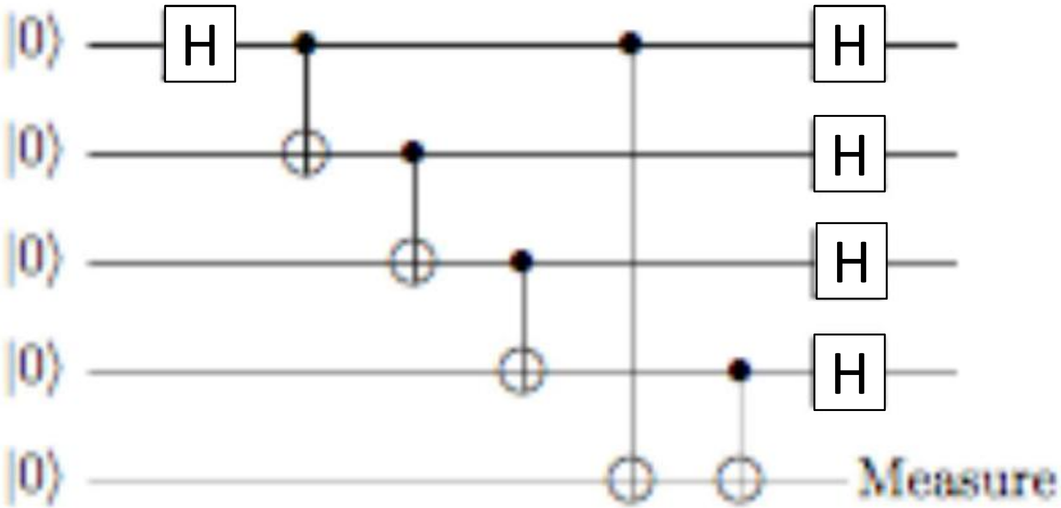



- The parity of the data qubits is mapped on the parity of the Shor state.
- An error in preparation of the Shor state can propagate.

J. Preskill, Fault-tolerant quantum computation, 1997.

# TOWARDS A SOLUTION

Idea N3: verification of ancillas




$$H = \frac{1}{\sqrt{2}}\left(X + Z\right)$$

- Parity measurement is launched if the 5th qubit is measured in 0.
- Otherwise repeat the preparation.

J. Preskill, Fault-tolerant quantum computation, 1997.

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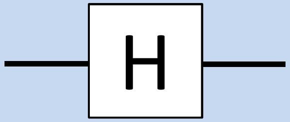
# A UNIVERSAL SET OF LOGICAL GATES

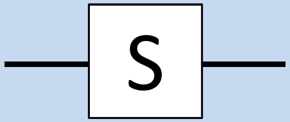
## Single qubit gates

Hadamard



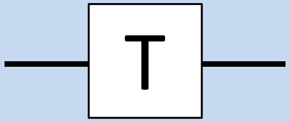

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(X+Z)$$

Phase




$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \exp(i\pi/4)\exp(-i\pi/4Z) = T^2$$

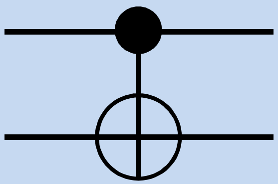
$\pi/8$




$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \exp(i\pi/8)\exp(-i\pi/8Z)$$

## Two qubit gate

C-NOT



$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

# BACK TO STABILIZER CODES: EXAMPLE OF STEANE

## Logical operators and action of gates:

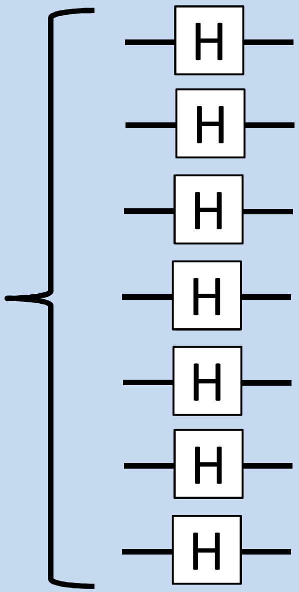
Logical operators: $\bar{Z} = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7$, $\bar{X} = X_1 X_2 X_3 X_4 X_5 X_6 X_7$, $\bar{Y} = i\bar{X}\bar{Z}$

Logical Hadamard: $\bar{H}\bar{Z}\bar{H} = \bar{X}$, $\bar{H}\bar{X}\bar{H} = \bar{Z}$, $\bar{H}\bar{Y}\bar{H} = -\bar{Y}$

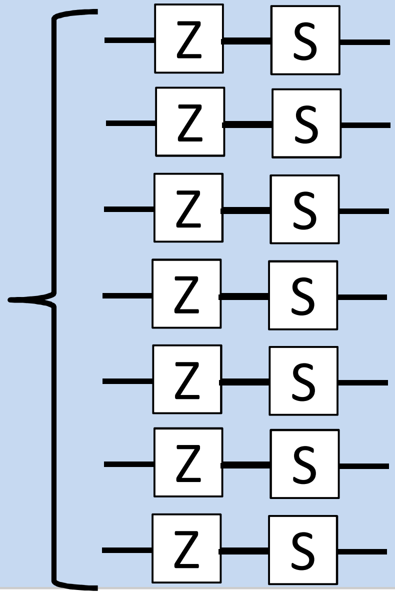
Logical Phase: $\bar{S}\bar{Z}\bar{S} = \bar{Z}$, $\bar{S}\bar{X}\bar{S} = \bar{Y}$, $\bar{H}\bar{Y}\bar{H} = -\bar{X}$

## Fault-tolerant choice:

$\bar{H} =$

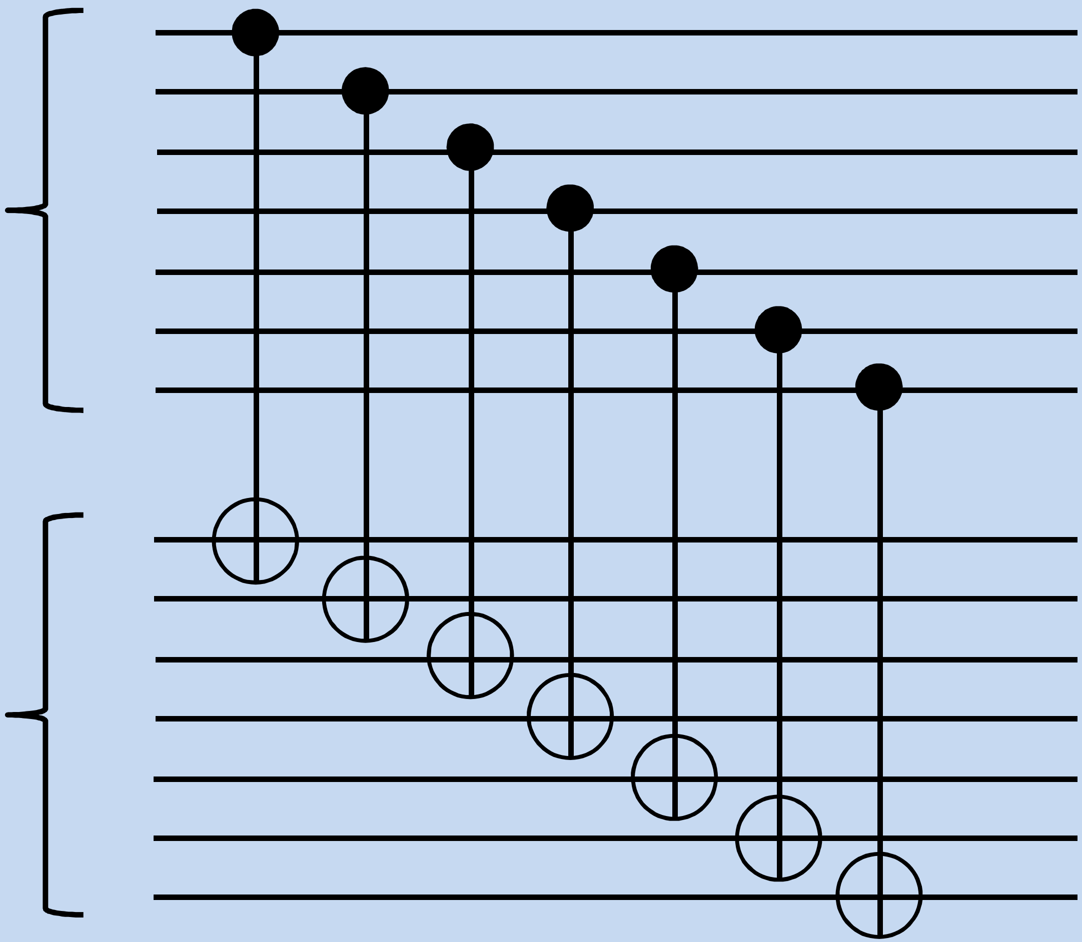



$\bar{S} =$



# BACK TO STABILIZER CODES: EXAMPLE OF STEANE

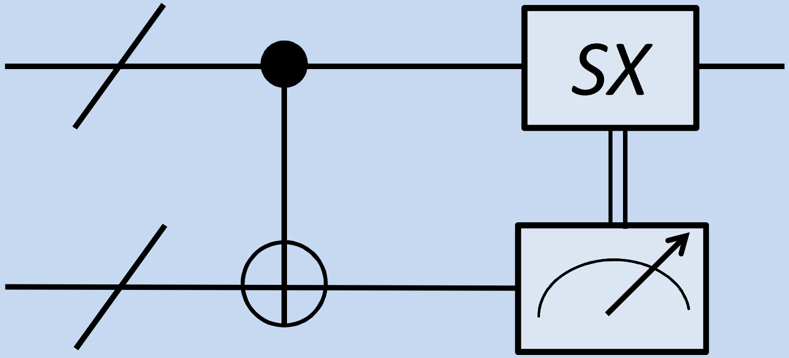
## Fault-tolerant C-NOT:



# BACK TO STABILIZER CODES: EXAMPLE OF STEANE

## Fault-tolerant T-gate:

$$\frac{\left(|0\rangle_L + \exp(i\pi/4)|1\rangle_L\right)}{\sqrt{2}}$$




$|\psi\rangle$ → $T|\psi\rangle$

Requires fault-tolerant preparation/distillation of magic state

$$\frac{\left(|0\rangle_L + \exp(i\pi/4)|1\rangle_L\right)}{\sqrt{2}}$$

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# CONCATENATION OF CODES

## Concatenation of codes $C_1$ (size $n_1$) and C2 (size $n_2$)

We construct a code of size $n_1 n_2$, where each qubit of $C_2$ is replaced by a block of $n_1$ qubits encoded in $C_1$.

## Higher order QEC by concatenation

| Level of concatenation | Error probability |
|---|---|
| Physical qubits | $\varepsilon_0 = p$ |
| 1st encoded level | $\varepsilon_1 = cp^2 = c^{-1}(cp)^2$ (*) |
| 2nd encoded level | $\varepsilon_2 = c(cp^2)^2 = c^{-1}(cp)^{2^2}$ |
| ⋮ | ⋮ |
| r'th encoded level | $\varepsilon_r = c(\varepsilon_{r-1})^2 = c^{-1}(cp)^{2^r}$ |

(*) For the Steane code $c \approx 10^4$

# THRESHOLD THEOREM

A quantum circuit containing f(n) gates may be simulated with probability of error at most $\varepsilon$ using

$$\mathcal{O}\left(f(n)\text{poly}\left[\log(f(n)/\varepsilon)\right]\right)$$

gates on hardware whose components fail with probability at most p, provided that $p<p_{th}$ , and given **reasonable assumptions** on the noise.

# TOWARDS AN ERROR-CORRECTED QUBIT

Three main strategies for implementing a logical qubit:

- A register of physical qubits with full gate operations
- A fabric of physical qubits with nearest neighbor gates
- A superconducting resonator with non-linear drives, non-linear dissipation and photon parity monitoring. These services are provided by Josephson junctions.

Shor (1995)
Steane (1996)
Gottesman, Kitaev, Preskill (2001)
Kitaev (2006)
M.M. et al. (2014)

# OUTLINE

- ❑ Classical vs quantum error correction
- ❑ Theory of quantum error correction
- ❑ Stabilizer formalism
- ❑ Fault-tolerant QEC
- ❑ Fault-tolerant logical gates
- ❑ Concatenation and threshold theorem
- ❑ A brief introduction to surface codes
- ❑ A brief introduction to continuous variable codes

# SURFACE CODE: ENCODING

Stabilizers:

$$A_s = \prod_{j \in \text{star } s} X_j \qquad B_p = \prod_{j \in \partial(p)} Z_j$$

Stabilizer (protected) subspace:

$$\mathcal{L} = \left\{ |\xi\rangle \middle| A_s |\xi\rangle = B_p |\xi\rangle = |\xi\rangle \right\}$$

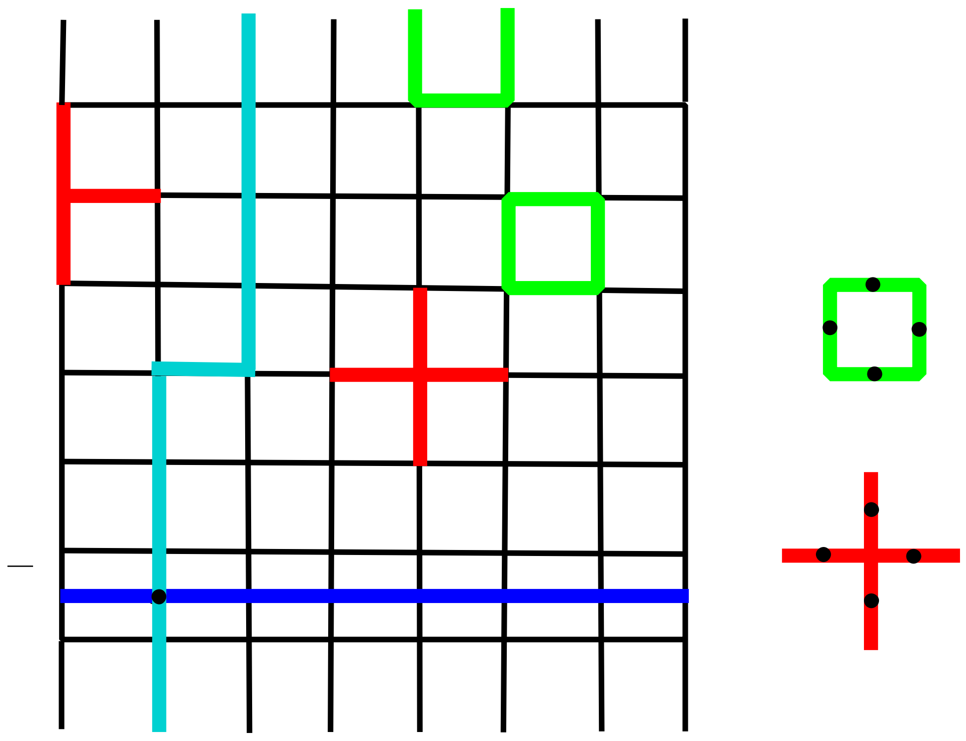
Number of qubits:

$$n = L^2 + (L-1)^2$$

Number of independent stabilizers:

$$r = 2L(L-1)$$

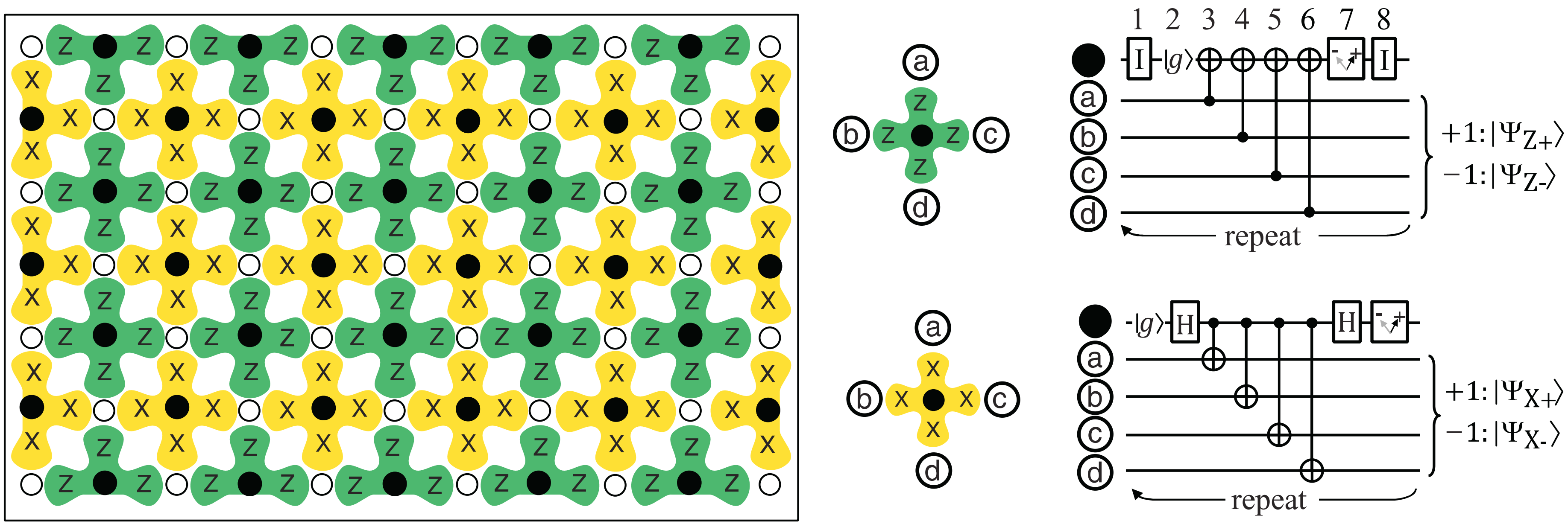
Encoded space: $\text{Dim}(\mathcal{L}) = 2^{n-r} = 2$



Logical operators:

$$Z_L = \prod_{j \in C_z} Z_j \qquad X_L = \prod_{j \in C'_x} X_j$$

# SURFACE CODE: PROTECTION




Black discs: measurement qubits
White discs: data qubits

Fowler et al. (2012)

# INCREASING THE NUMBER OF LOGICAL QUBITS

Turning off some stabilizer measurements:

Number of qubits: $n = L^2 + (L-1)^2$

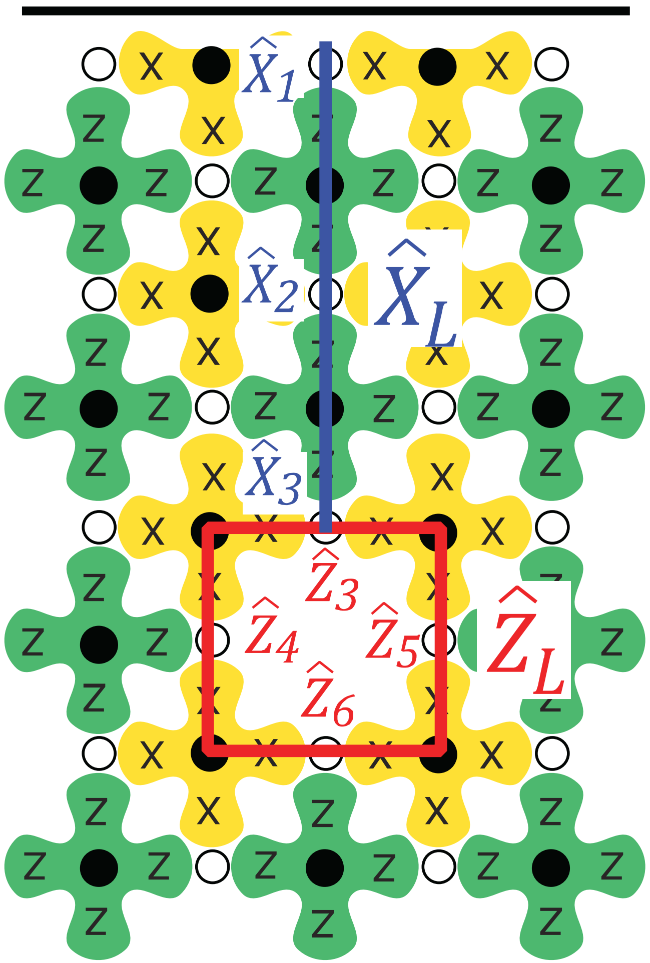
Number of independent stabilizers:

$$r = 2L(L-1) - 1$$

Encoded space: $\mathrm{Dim}(\mathcal{L}) = 2^{n-r} = 2^2$

How about code distance: 4 at max.

Solution?




Fowler et al. (2012)

# INCREASING THE NUMBER OF LOGICAL QUBITS

Turning off some stabilizer measurements:

Number of qubits: $n = L^2 + (L-1)^2$

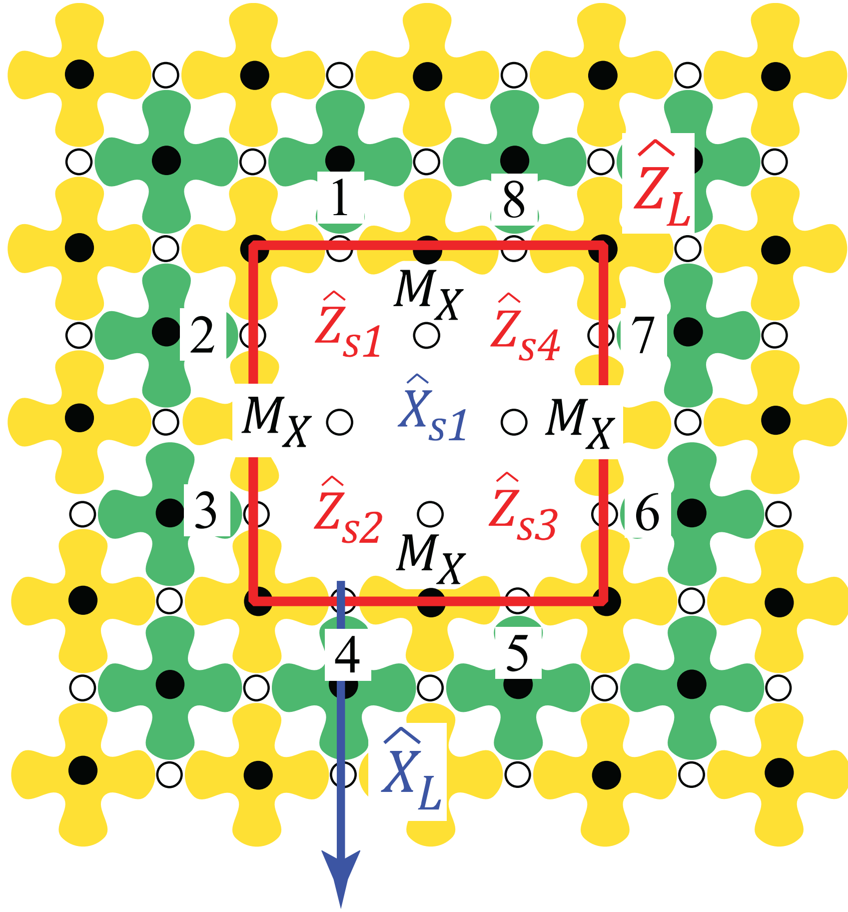
Number of independent stabilizers:

$$r = 2L(L-1) - 1$$

Encoded space: $\mathrm{Dim}\left(\mathcal{L}\right) = 2^{n-r} = 2^2$

How about code distance: 4 at max.

Solution: larger deffect




Fowler et al. (2012)

# QUANTUM COMPUTATION WITH SURFACE CODES

1. Initialization: projectively measure logical operators
2. Moving qubits around (modifying stabilizer constraints)
3. CNOT gates between qubits (braiding operations)
4. Hadamard gate (modifying stabilizer constraints and physical Hadamard on a set of qubits)
5. S and T gates (magic state distillation and teleportation)

Fowler et al. (2012)

# SURFACE CODE: ESTIMATED PERFORMACES

<u>Error model:</u>

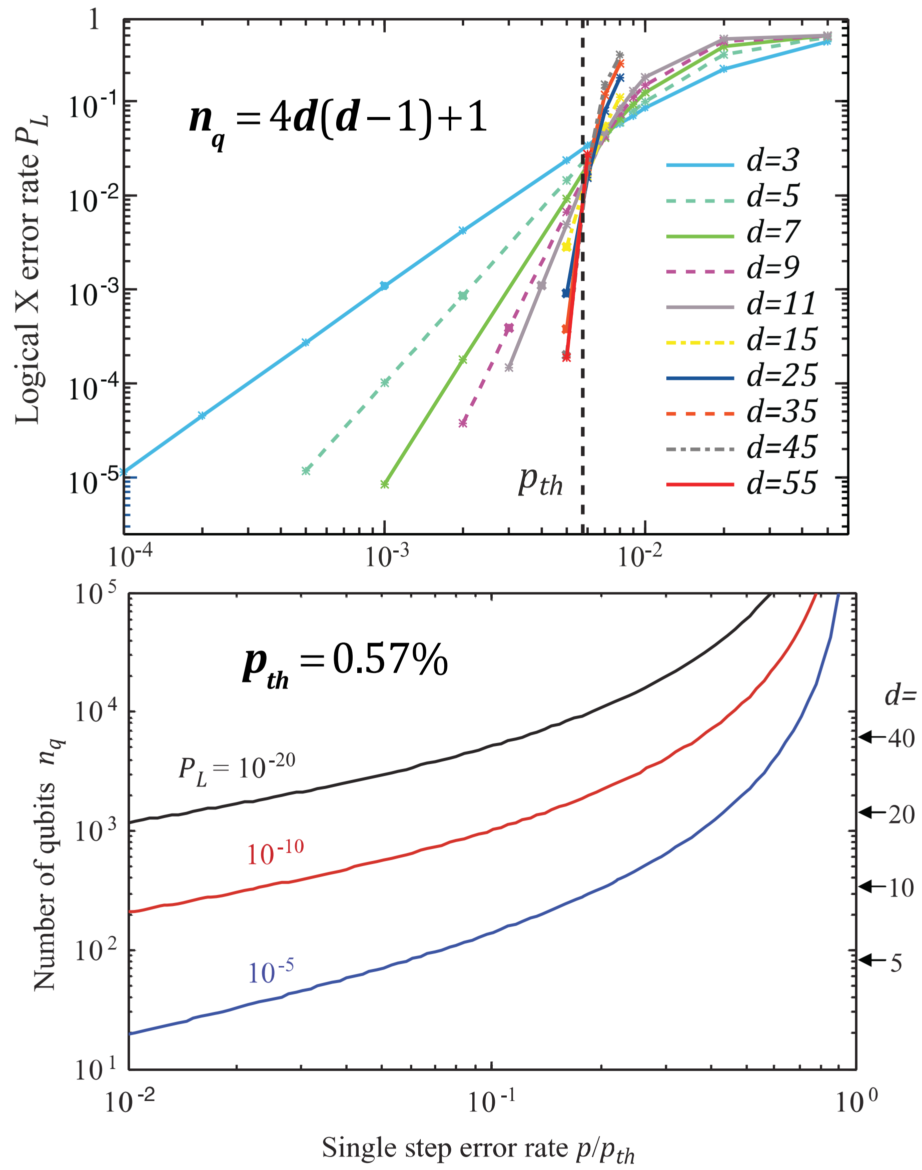
1- attemting to perform a data qubit identitiy, but instead performing single-qubit X, Y, Z, each with proba p/3.

2- attemting to initilize |g> but instead preparing |e> with proba p.

3- attempting to perform H, but in addition one single-qubit operation X, Y, Z with proba p/3.

4- measurement error with proba p.

5- attempting to perform measure qubit-data qubit CNOT, but instead one of the two qubit operations $X_{1,2}$, $Y_{1,2}$, $Z_{1,2}$, $X_1X_2$, $X_1Y_2$, $X_1Z_2$, $Y_1X_2$, $Y_1Y_2$, $Y_1Z_2$, $Z_1X_2$, $Z_1Y_2$, $Z_1Z_2$ with proba p/15.




Fowler et al. (2012)

# OUTLINE

- ❑ Classical vs quantum error correction
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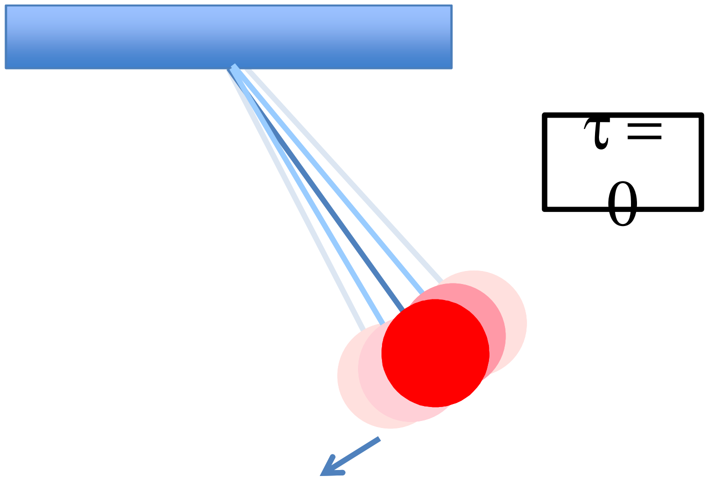
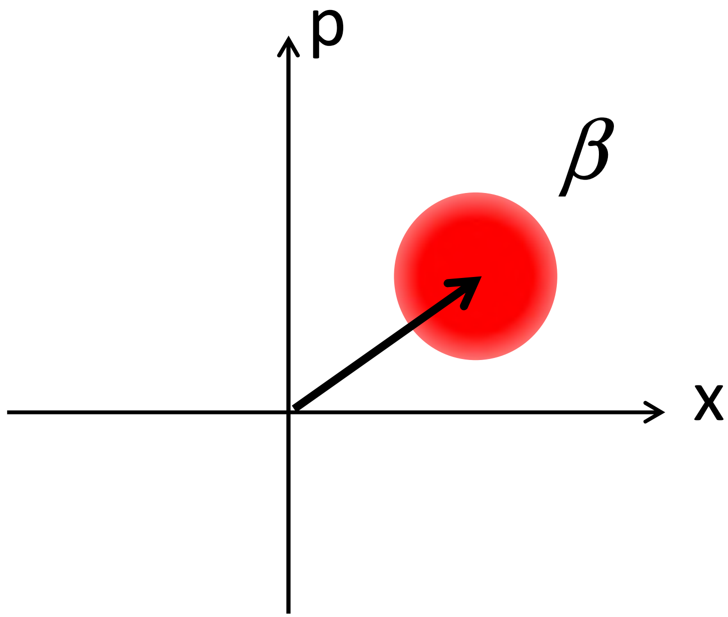
# QUANTUM HARMONIC OSCILLATOR AND COHERENT STATES

Using classical control (e.g. laser, force), one can only make coherent displacements

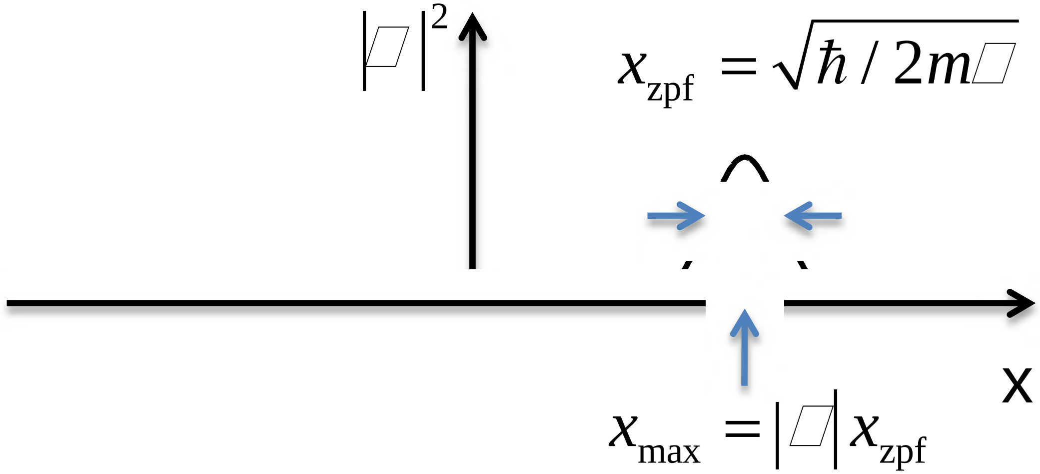


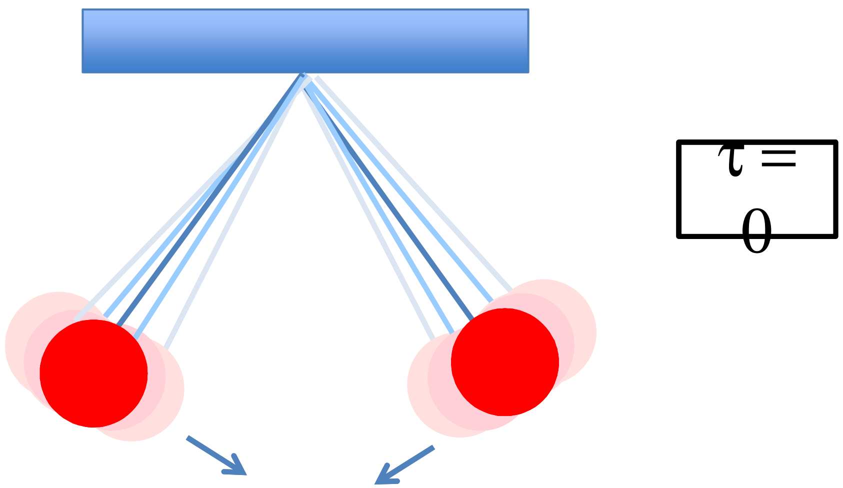

Glauber (coherent) state

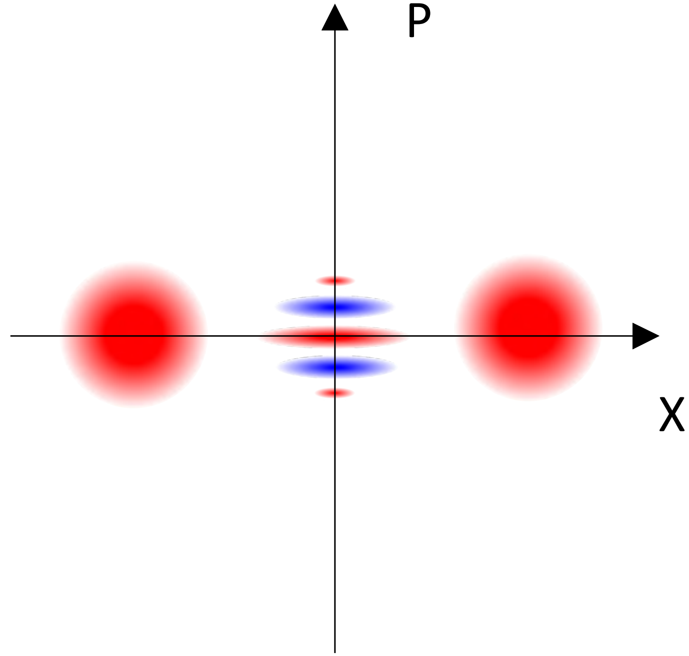
$$|\beta\rangle = e^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle$$

$$\hat{a}|\beta\rangle = \beta|\beta\rangle$$



# SCHRÖDINGER CAT STATE FOR A HARMONIC OSCILLATOR



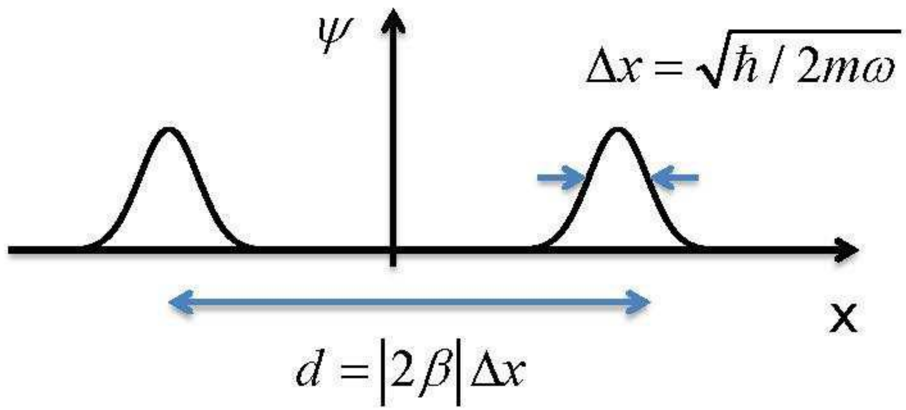




Cat state of an oscillator

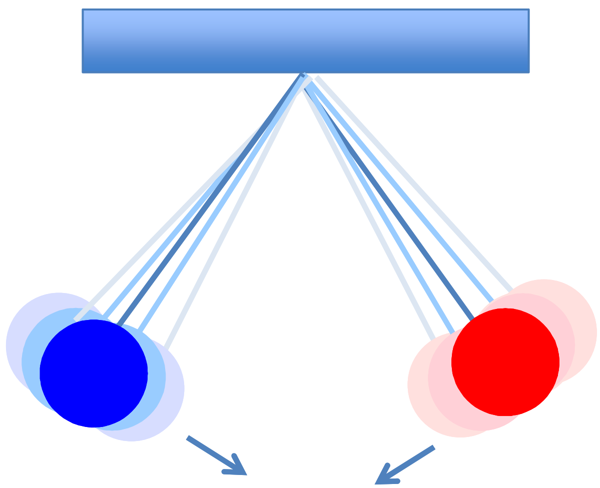
Wigner function W(β)

$$\left|C_\beta^+\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\beta\right\rangle + \left|-\beta\right\rangle\right)$$

$$= \frac{1}{\sqrt{\cosh|\beta|^2}} \sum \frac{\beta^{2n}}{\sqrt{(2n)!}}\left|2n\right\rangle$$

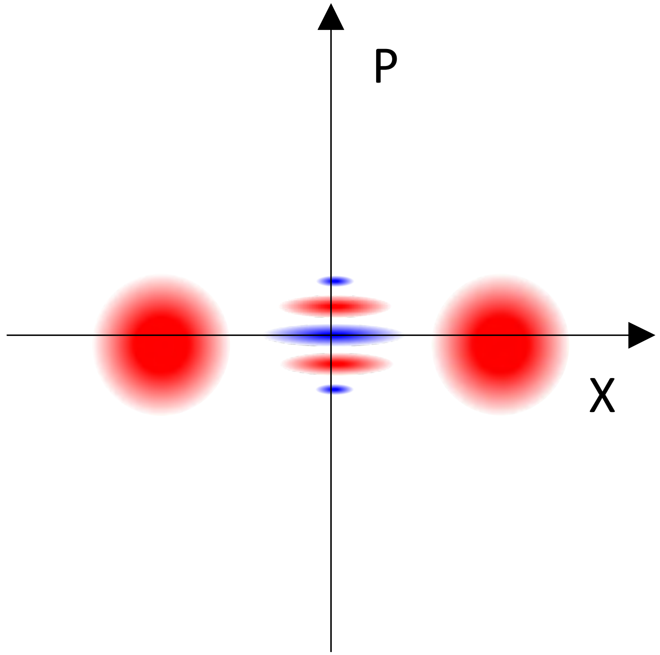


# SCHRÖDINGER CAT STATE FOR A HARMONIC OSCILLATOR



$\tau = 0$

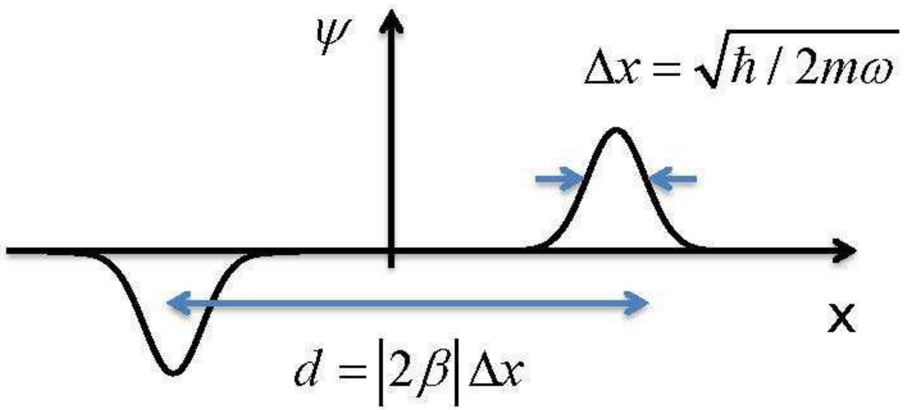
Cat state of an oscillator



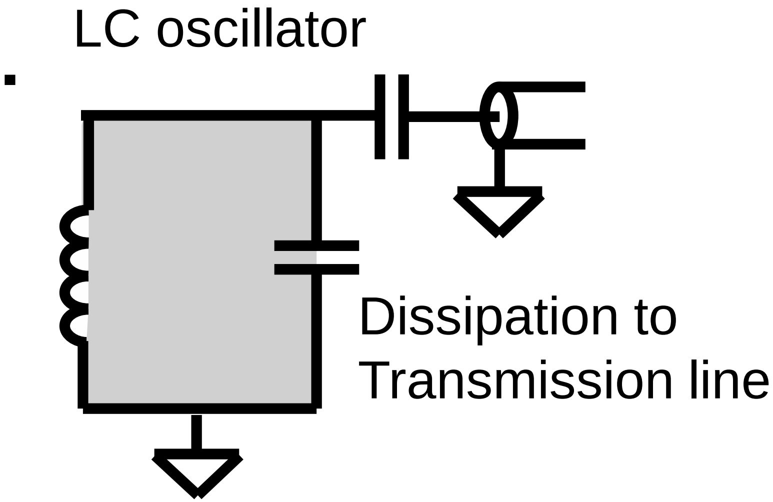

Wigner function W(β)

$$\left|C_\beta^-\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\beta\right\rangle - \left|-\beta\right\rangle\right)$$

$$= \frac{1}{\sqrt{\sinh|\beta|^2}} \sum \frac{\beta^{(2n+1)}}{\sqrt{(2n+1)!}} \left|2n+1\right\rangle$$



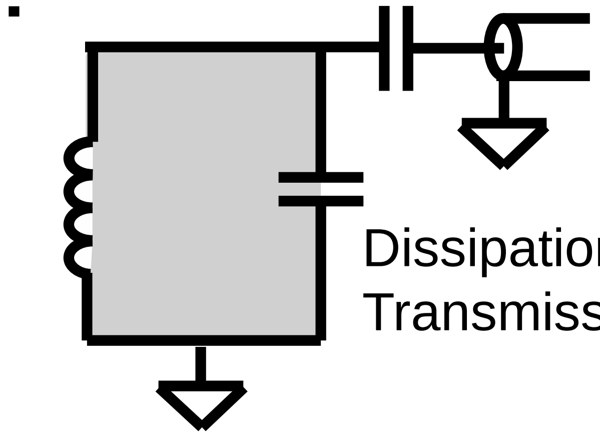
# PHOTON LOSS: MAJOR DECAY CHANNEL OF A H.O.




$$\frac{d}{dt}\rho = \kappa D[\mathbf{a}]\rho,$$

$$D[\mathbf{a}]\rho = \mathbf{a}\rho\mathbf{a}^{\dagger} - \frac{1}{2}\mathbf{a}^{\dagger}\mathbf{a}\rho - \frac{1}{2}\rho\mathbf{a}^{\dagger}\mathbf{a}.$$

# PHOTON LOSS: MAJOR DECAY CHANNEL OF A H.O.




$$\frac{d}{dt}\rho = \kappa D[\mathbf{a}]\rho,$$

$$D[\mathbf{a}]\rho = \mathbf{a}\rho\mathbf{a}^{\dagger} - \frac{1}{2}\mathbf{a}^{\dagger}\mathbf{a}\rho - \frac{1}{2}\rho\mathbf{a}^{\dagger}\mathbf{a}.$$

Formulation with error channels:

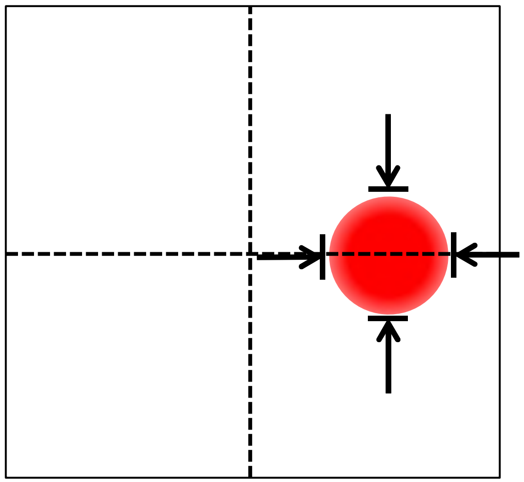
$$\rho_{\delta t} = \mathcal{E}(\rho_0) = \sum_{l=0}^{\infty} \mathbf{E}_l \rho_0 \mathbf{E}_l^{\dagger}, \quad \mathbf{E}_l = \sqrt{\frac{\left(1 - e^{-\kappa\delta t}\right)^l}{l!}}\, e^{-\frac{\kappa\delta t}{2}\boldsymbol{a}^{\dagger}\boldsymbol{a}}\boldsymbol{a}^l$$

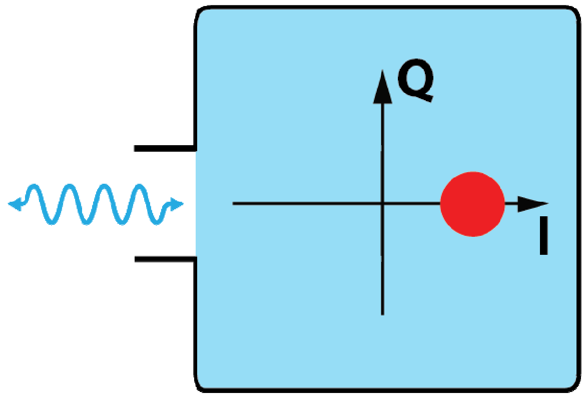
I.L. Chuang et al., PRA 56, 1997.

# CAT PUMPING

driven damped harmonic oscillator :

$$\mathbf{H} = i\varepsilon_1(\mathbf{a} - \mathbf{a}^\dagger), \quad \mathbf{L} = \sqrt{\kappa_1}\,\mathbf{a}$$





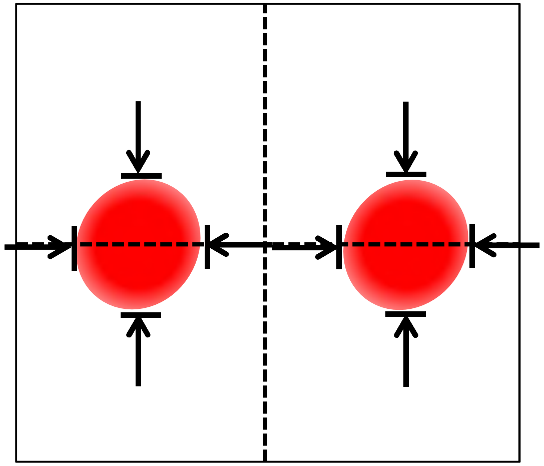

Steady state $|\alpha\rangle, \qquad \alpha = 2\varepsilon_1 / \kappa_1$

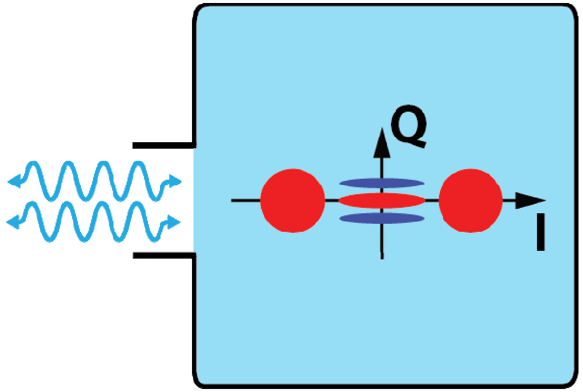
# CAT PUMPING

New type of drive :
2-photon exchange with the environment :

$$\mathbf{H} = i\varepsilon_2(\mathbf{a}^2 - \mathbf{a}^{\dagger 2}), \quad \mathbf{L} = \sqrt{\kappa_2}\mathbf{a}^2$$

$$|0_L\rangle = |\mathcal{C}_\alpha^+\rangle$$
$$|1_L\rangle = |\mathcal{C}_\alpha^-\rangle$$



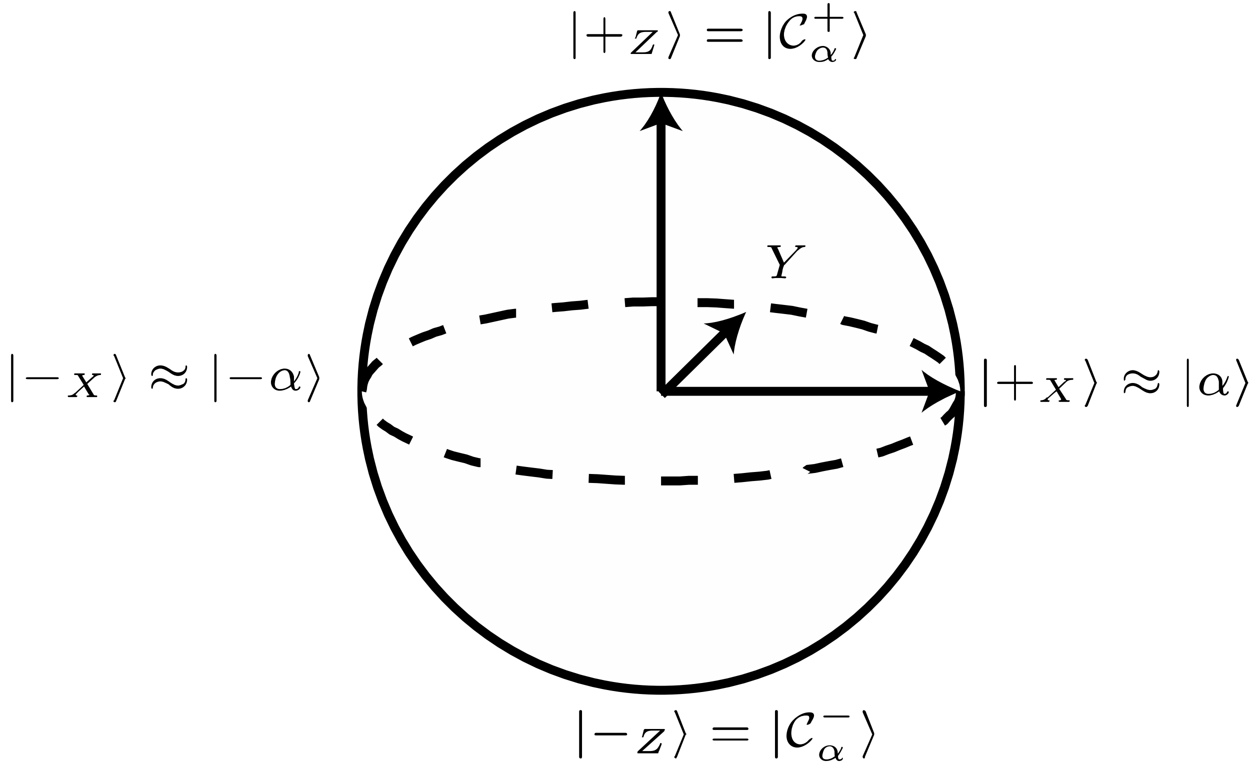



Asymptotic 2D-manifold

$$\mathcal{M}_{2,\alpha} = \mathrm{span}\{|\mathcal{C}_\alpha^+\rangle, |\mathcal{C}_\alpha^-\rangle\} \qquad \alpha = \sqrt{2\varepsilon_2/\kappa_2}$$

Leghtas *et al.* Science (2015)

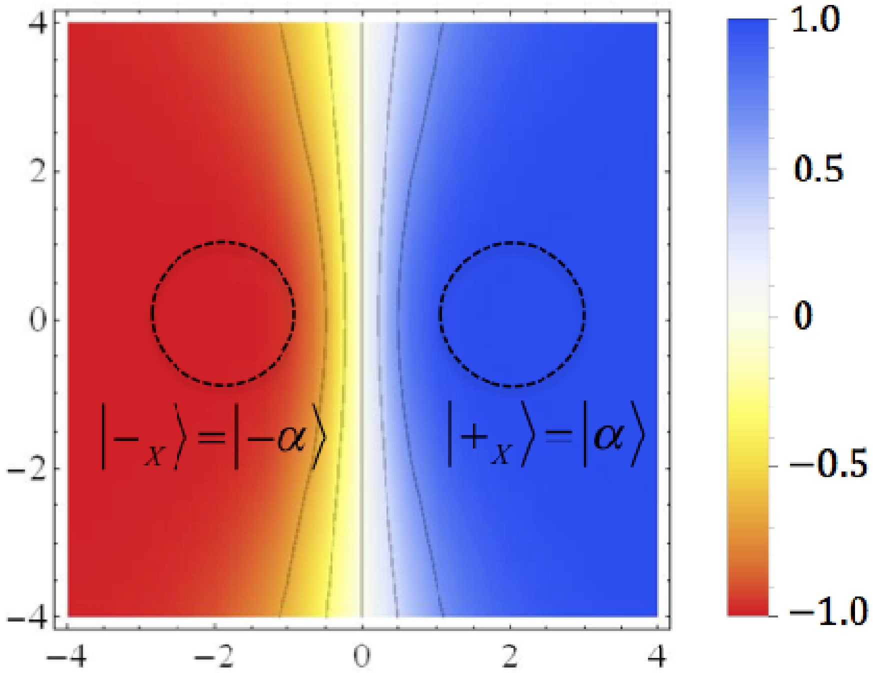
# CHOICE OF QUBIT BASIS

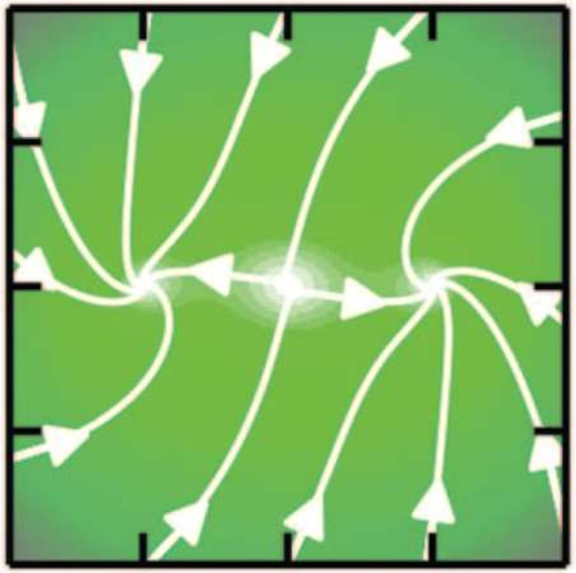



$$|+_z\rangle = |C_\alpha^+\rangle = N_+(|\alpha\rangle + |-\alpha\rangle) = \sum c_{2n}|2n\rangle$$

$$|-_z\rangle = |C_\alpha^-\rangle = N_-(|\alpha\rangle - |-\alpha\rangle) = \sum c_{2n+1}|2n+1\rangle$$

# PUMPED CATS: A QUBIT WITHOUT PHASE-FLIPS



$\mathrm{Tr}(\sigma_X^L \rho_\infty)$ for $\rho_0 = |\beta\rangle\langle\beta|$

Phase-flip errors induced by reasonable (local in the phase space) errors are suppressed exponentially in $|\alpha|^2$ .

# PUMPED CATS: A QUBIT WITHOUT PHASE-FLIPS

An arbitrary error channel on a harmonic oscillator:

$$\rho \to \mathbb{E}\left(\rho\right) = \sum_k \boldsymbol{E}_k(\boldsymbol{a},\boldsymbol{a}^\dagger)\rho \boldsymbol{E}_k(\boldsymbol{a},\boldsymbol{a}^\dagger)^\dagger$$

A general identity:

$$\boldsymbol{E}(\boldsymbol{a},\boldsymbol{a}^\dagger)\Pi_{\mathcal{M}_{2,\alpha}} = \boldsymbol{F}^I(\boldsymbol{a}^2,\boldsymbol{a}^{\dagger 2},\boldsymbol{a}^\dagger \boldsymbol{a})\Pi_{\mathcal{M}_{2,\alpha}} + \boldsymbol{F}^{X,\alpha}(\boldsymbol{a}^2,\boldsymbol{a}^{\dagger 2},\boldsymbol{a}^\dagger \boldsymbol{a})\sigma_X^L$$

Where:

$$\mathcal{M}_{2,\alpha} = \mathrm{span}\left\{\left|\alpha\right\rangle,\left|-\alpha\right\rangle\right\}, \qquad \sigma_X^L = \left|C_\alpha^+\right\rangle\left\langle C_\alpha^-\right| + \left|C_\alpha^-\right\rangle\left\langle C_\alpha^+\right|$$

# PUMPED CATS: A QUBIT WITHOUT PHASE-FLIPS

Correctability criteria for the cat-code:

$$\Pi_{\mathcal{M}_{2,\alpha}} \boldsymbol{F}_j^\dagger \boldsymbol{F}_k \Pi_{\mathcal{M}_{2,\alpha}} = c_{jk} \Pi_{\mathcal{M}_{2,\alpha}}$$

An appropriate basis for the error operators:

$$\boldsymbol{F}(\boldsymbol{a}^2, \boldsymbol{a}^{\dagger 2}, \boldsymbol{a}^\dagger \boldsymbol{a}) \Pi_{\mathcal{M}_{2,\alpha}} = \int_{\mathrm{Re}(\beta)>0} d^2\beta\, u^\alpha(\beta)(\boldsymbol{D}_\beta + \boldsymbol{D}_{-\beta}) \Pi_{\mathcal{M}_{2,\alpha}}$$

It is enough to illustrate the correctability of the symmetric displacement operators.

Gottesman, Kitaev, Preskill, PRA, 2001.

# PUMPED CATS: A QUBIT WITHOUT PHASE-FLIPS

Under the condition of small displacements: $|\beta| \leq R_{\max}$

$$\Pi_{\mathcal{M}_{2,\alpha}}(\boldsymbol{D}_{\beta_1} + \boldsymbol{D}_{-\beta_1})^{\dagger}(\boldsymbol{D}_{\beta_2} + \boldsymbol{D}_{-\beta_2})\Pi_{\mathcal{M}_{2,\alpha}} = c_{\beta_1,\beta_2}\Pi_{\mathcal{M}_{2,\alpha}} + \varepsilon\sigma_Z^L$$

Where: $\varepsilon \leq 4e^{-2(|\alpha| - R_{\max})^2}$

Furthermore pumping is the correction operation:

$$\mathbb{R}_{\text{pump}} \circ \mathbb{F}(\rho) = \rho + \mathcal{O}\left(e^{-c(|\alpha| - R_{\max})^2}\right)$$

# PUMPED CATS: A QUBIT WITHOUT PHASE-FLIPS

Example 1: photon-loss channel

$$\boldsymbol{E}_k = \sqrt{\frac{(1-e^{-\kappa\delta t})^k}{k!}} e^{-\frac{\kappa\delta t}{2}\boldsymbol{a}^\dagger \boldsymbol{a}} \boldsymbol{a}^k$$

Therefore

$$\boldsymbol{F}_{2k}\Pi_{\mathcal{M}_{2,\alpha}} = \boldsymbol{F}_{2k+1}\Pi_{\mathcal{M}_{2,\alpha}} = \alpha^{2k}\sqrt{\frac{(1-e^{-\kappa\delta t})^{2k}}{(2k)!}} e^{-\frac{\kappa\delta t}{2}\boldsymbol{a}^\dagger \boldsymbol{a}} \Pi_{\mathcal{M}_{2,\alpha}}$$

Leading to

$$\mathbb{R}_{\text{pump}} \circ \mathbb{F}(\rho) = \rho + \mathcal{O}\left(e^{-c|\alpha|^2 e^{-\kappa\delta t}}\right)$$

# PUMPED CATS: A QUBIT WITHOUT PHASE-FLIPS

Example 2: phase-noise due to dispersive coupling to a hot mode

$$\boldsymbol{H}_{\text{int}} = -\hbar\chi \boldsymbol{a}^\dagger \boldsymbol{a} \boldsymbol{b}^\dagger \boldsymbol{b} \quad \text{with} \quad \rho_b^s = \sum p_n |n\rangle\langle n|$$

Set of error operators:

$$\boldsymbol{F}_k = \sqrt{p_n}\, e^{i\chi n \delta t \boldsymbol{a}^\dagger \boldsymbol{a}}$$

Leading to

$$\mathbb{R}_{\text{pump}} \circ \mathbb{F}(\rho) = \rho + \mathcal{O}\left(\sum p_n e^{-c \max(|\alpha|\cos(\chi n \delta t), 0)^2}\right).$$

# TOWARDS FULL PROTECTION:

**Two approaches:**

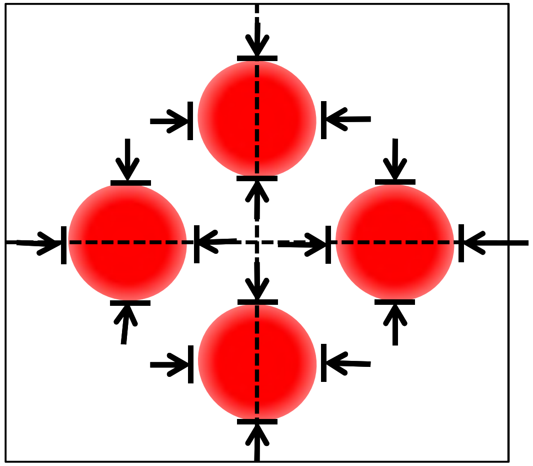
- Multi-component cats: 4-photon pumping for protection against photon annihilation operator…..
- Multi-mode cats: Protection against logical bit-flips. This is more general as it includes any remaining error channel: photon loss, thermal excitations, higher-order non-linearities….

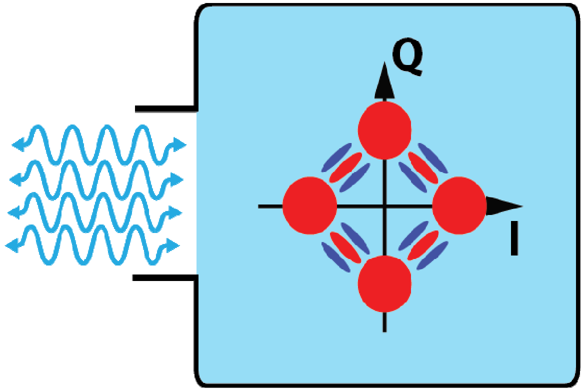
# CAT PUMPING

New type of drive :
4-photon exchange with the environment :

$$\mathbf{H} = i\varepsilon_4(\mathbf{a}^4 - \mathbf{a}^{\dagger 4}), \quad \mathbf{L} \propto \sqrt{\kappa_4}\,\mathbf{a}^4$$

$$|0_L\rangle = |\mathcal{C}_\alpha^+\rangle$$
$$|1_L\rangle = |\mathcal{C}_{i\alpha}^+\rangle$$






Asymptotic 4D-manifold

$$\mathcal{M}_{4,\alpha} = \mathrm{span}\{|\mathcal{C}_\alpha^+\rangle, |\mathcal{C}_{i\alpha}^+\rangle, |\mathcal{C}_\alpha^-\rangle, |\mathcal{C}_{i\alpha}^-\rangle\}$$

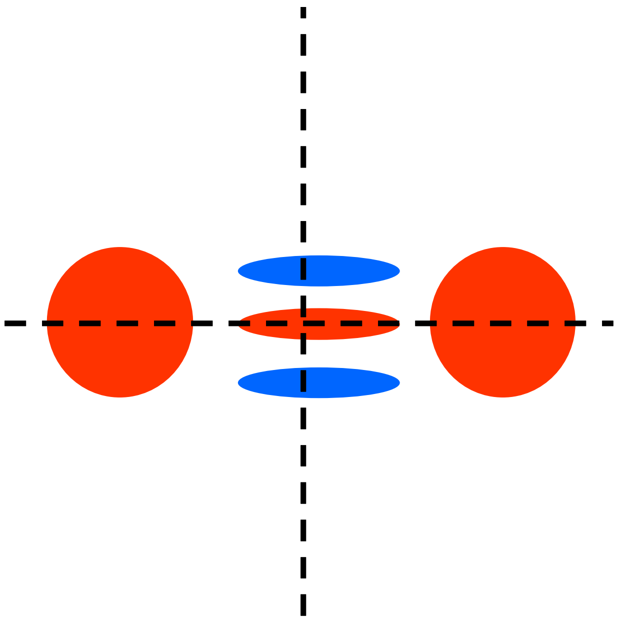
$$\alpha = \sqrt[4]{2\varepsilon_4 / \kappa_4}$$

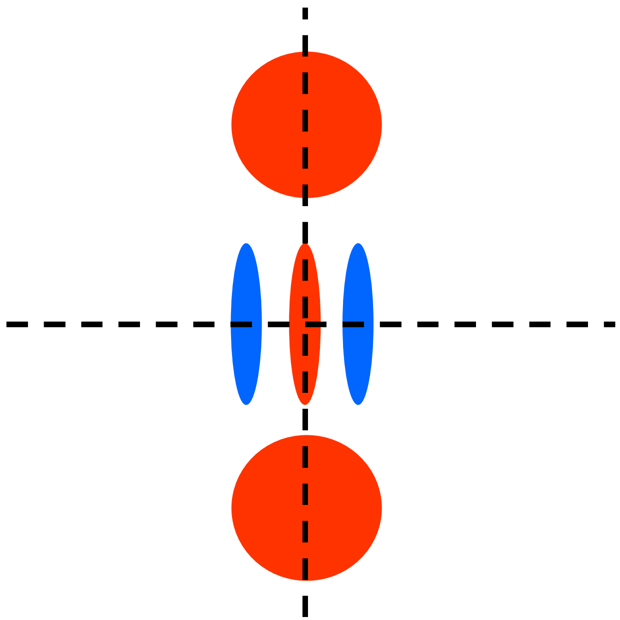
M.M. *et al.* NJP (2014),

# HARDWARE-EFFICIENT QUANTUM ERROR CORRECTION

**Idea:**

$$|0_L\rangle = |C_\alpha^+\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |-\alpha\rangle) \quad |1_L\rangle = |C_{i\alpha}^+\rangle = \frac{1}{\sqrt{2}}(|i\alpha\rangle + |-i\alpha\rangle)$$

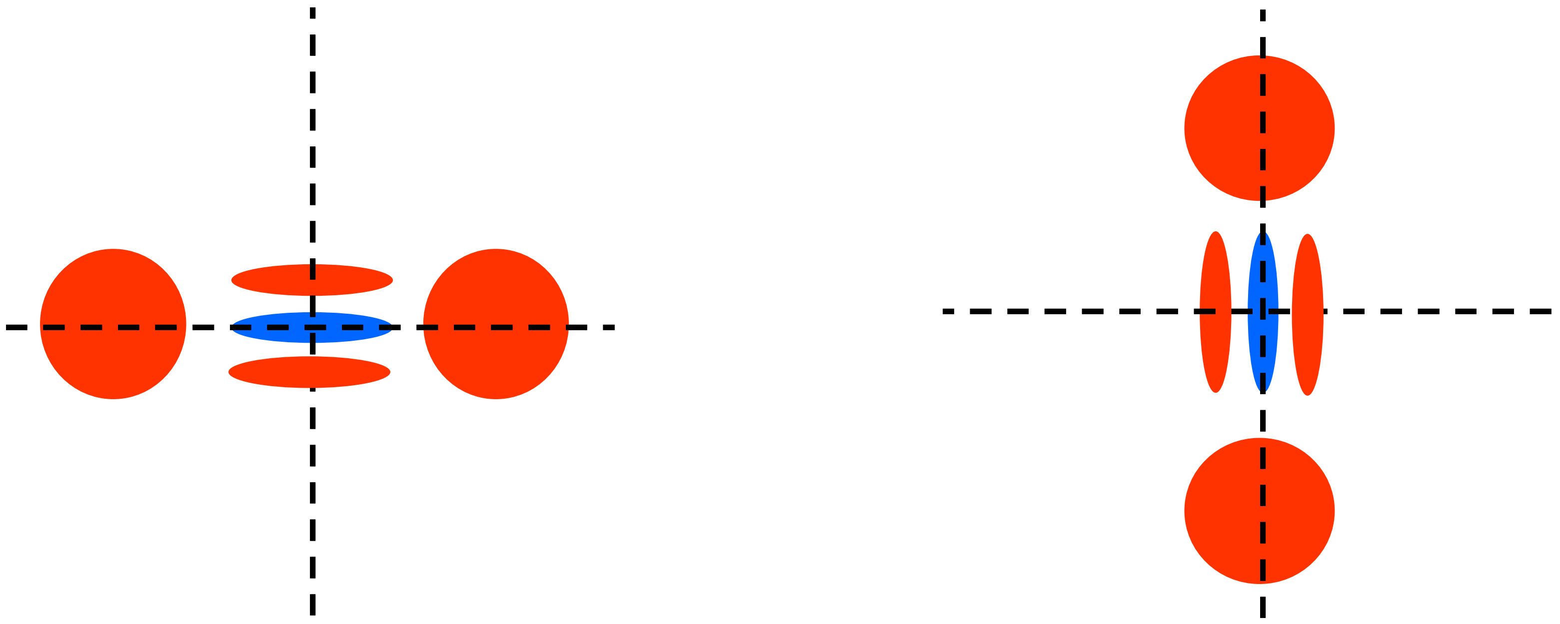




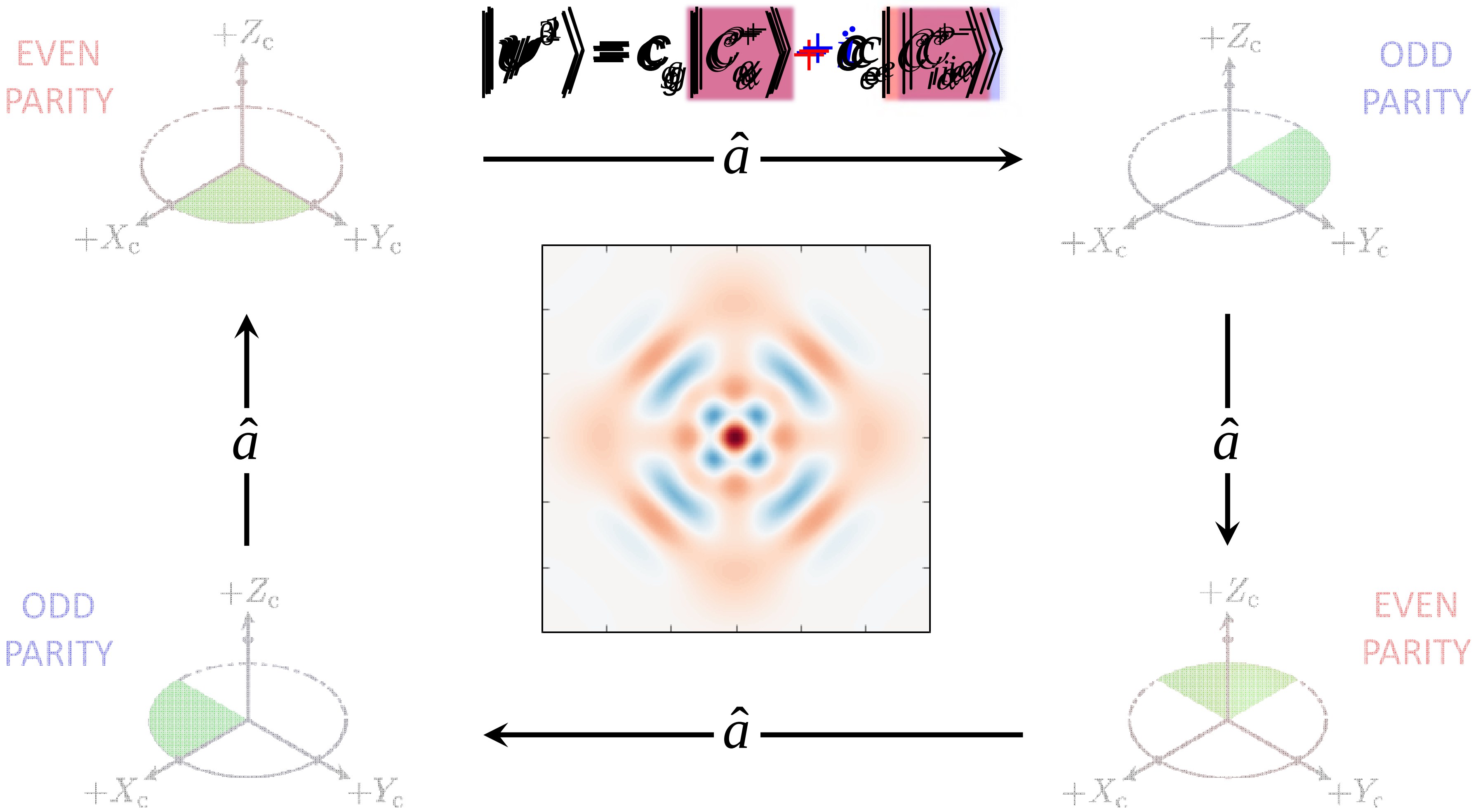
# HARDWARE-EFFICIENT QUANTUM ERROR CORRECTION

**Another possibility:**

$$|0_L\rangle = |C_\alpha^-\rangle = \frac{1}{\sqrt{2}}\left(|\alpha\rangle - |-\alpha\rangle\right) \qquad |1_L\rangle = |C_{i\alpha}^-\rangle = \frac{1}{\sqrt{2}}\left(|i\alpha\rangle - |-i\alpha\rangle\right)$$



# TO LIVE AND DIE IN A CAVITY




*Ofek et al., Nature 536, 441-445, 2016.

# QUANTUM COMPUTATION WITH CAT CODES

1. Half-protected logical gates through Zeno dynamics

2. Fault-tolerant photon number parity measurements

3. Higher-order codes and fully protected logical gates (ungoing)

M.M. et al., NJP, 2014
J. Cohen et al., PRL, 2017
S. Rosenblum et al., In press.