Quantum optics
of
many-body systems

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Lecture 3

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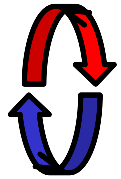
Previous lectures

Classical optics
light waves
material devices

light

matter

Atom optics
matter waves
light forces

Devices:
beam splitters, mirrors, diffraction gratings, cavities

Quantum optics
classical (hot) atoms ☹
quantum light states ☺

Quantum atom optics
☺ quantum atomic states
☹ classical light

☺ **Quantum optics of quantum gases** ☺

Further plan

- **Quantum atom optics (many-body)**

  second quantization of matter fields

  Quantum gases in optical lattices

  superfluid, Mott insulator states

- **Quantum measurements**

  quantum trajectories and conditional evolution

  quantum nondemolition (QND) measurements

Second quantization: Quantum optics

Lecture 1: Quantization of the electromagnetic field

Classical Maxwell equation $$\nabla^2 \mathbf{E}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = 0$$

Classical modes and quantum operators:

$$E_x(z,t) = \sum_j \mathcal{E}_j (a_j e^{-i\omega_j t} + a_j^\dagger e^{i\omega_j t}) \sin k_j z$$

Energy: $$\mathcal{H} = \hbar \sum_j \omega_j \left(a_j^\dagger a_j + \frac{1}{2} \right)$$

$$[a_j, a_{j'}^\dagger] = \delta_{jj'}$$

$$[a_j, a_{j'}] = [a_j^\dagger, a_{j'}^\dagger] = 0$$

Second quantization: Quantum atom optics

Quantization of the matter field $\Psi(\mathbf{r},t)$ is an operator

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t)=\left(-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r},t)\right)\Psi(\mathbf{r},t)$$

Bosons: $[\Psi(\mathbf{r},t),\Psi^\dagger(\mathbf{r}',t)]=\delta(\mathbf{r}-\mathbf{r}')$

$\Psi(\mathbf{r})|0>=0$

Fermions: $[\Psi(\mathbf{r},t),\Psi^\dagger(\mathbf{r}',t)]_+=\delta(\mathbf{r}-\mathbf{r}')$

Hamiltonian

$$H=\int d^3\mathbf{r}\Psi^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r},t)\right)\Psi(\mathbf{r})$$

Connection to the wave function in the 1st quantization

$$|\phi_N\rangle=\frac{1}{\sqrt{N!}}\int d^3\mathbf{r}_1\mathbf{r}_2...\mathbf{r}_N\phi_N(\mathbf{r}_1,\mathbf{r}_2,..\mathbf{r}_N)\Psi^\dagger(\mathbf{r}_N)...\Psi^\dagger(\mathbf{r}_2)\Psi^\dagger(\mathbf{r}_1)|0>$$

Classical modes and quantum operators

$$\Psi(\mathbf{r},t) = \sum_n \hat{c}_n(t)\varphi_n(\mathbf{r})$$

Energy of oscillators

$$H = \sum_n E_n \hat{c}_n^\dagger \hat{c}_n$$

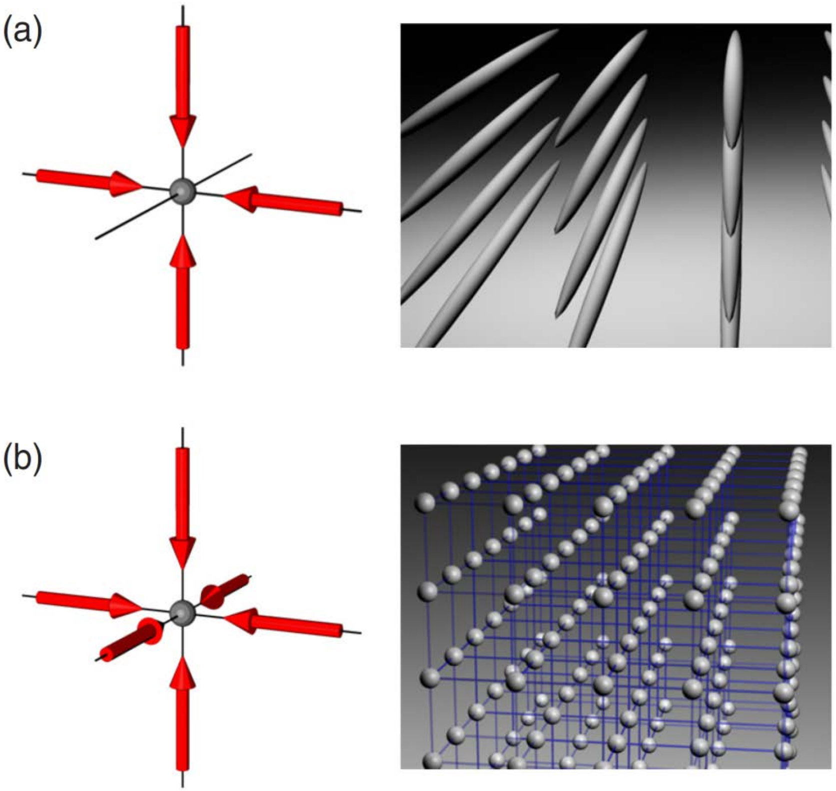
Bosons: $[\hat{c}_n, \hat{c}_m^\dagger] = \delta_{nm}$ fermions: $[\hat{c}_n, \hat{c}_m^\dagger]_+ = \delta_{nm}$

$$\hat{c}_n|N_n\rangle = \sqrt{N_n}|N_n - 1\rangle \qquad \hat{c}_n^\dagger|N_n\rangle = \sqrt{N_n + 1}|N_n + 1\rangle$$

Matter wave:

$$\varphi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{kr}}$$

Atoms in optical lattices



M. Greiner

A particle moving in a periodic potential *V*(*x*)

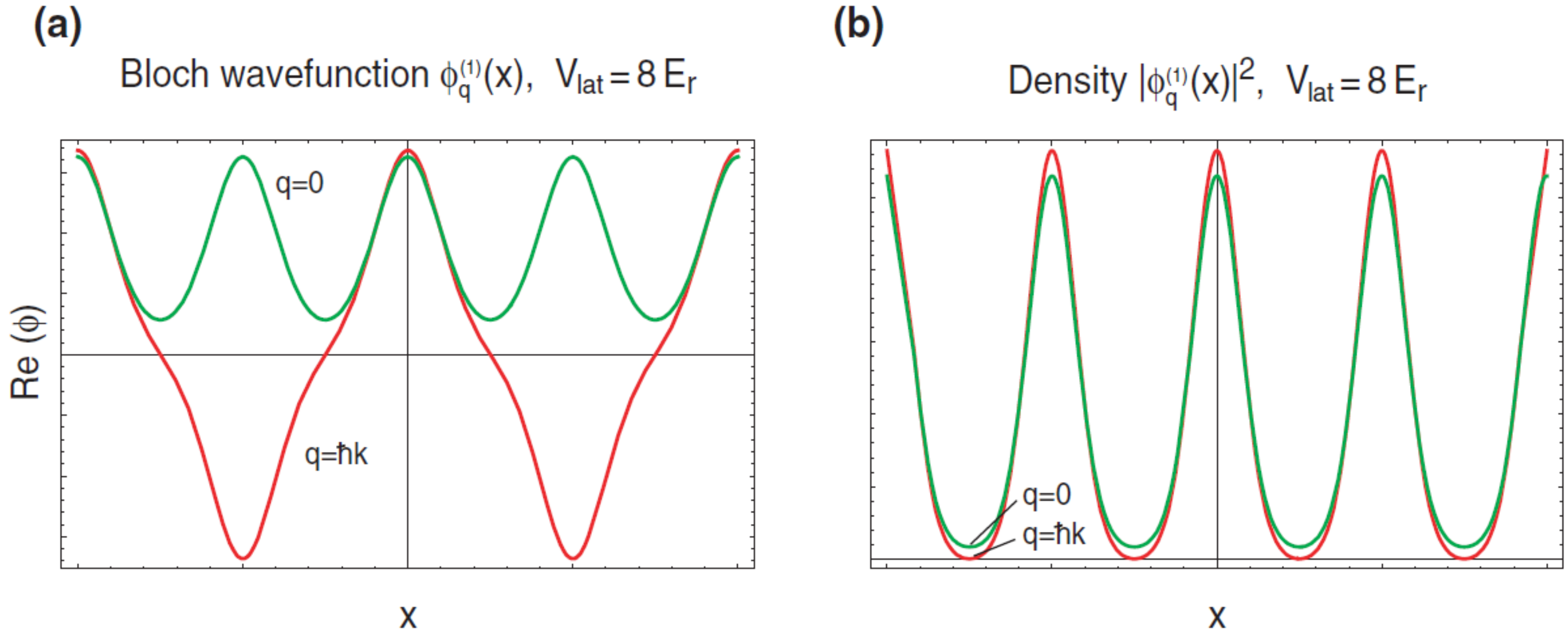
Schrödinger equation: $$H\phi_q^{(n)}(x) = E_q^{(n)}\phi_q^{(n)}(x)$$

$$H = \frac{1}{2m}\hat{p}^2 + V(x)$$

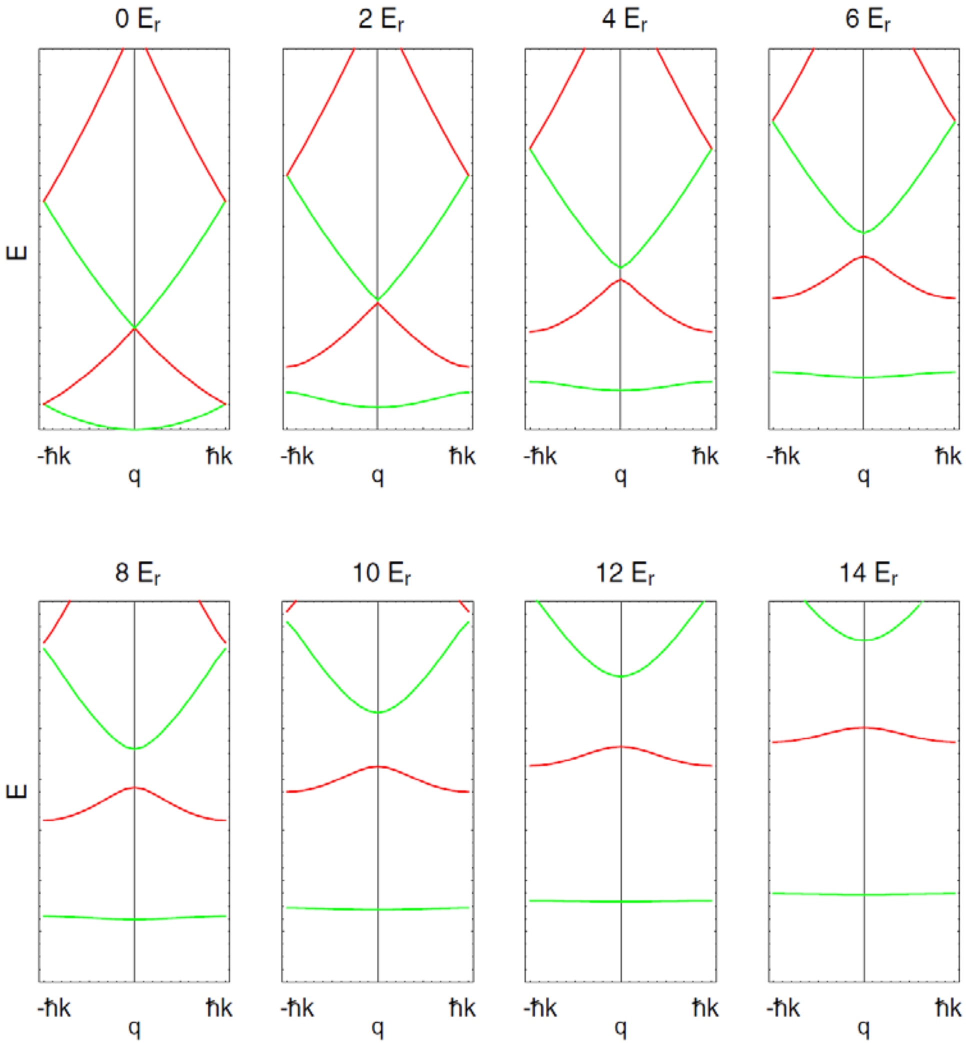
Solution: Bloch waves (delocalized in space)

$$\phi_q^{(n)}(x) = e^{iqx/\hbar} \cdot u_q^{(n)}(x)$$

$$E_r = \hbar^2 k^2/2m$$

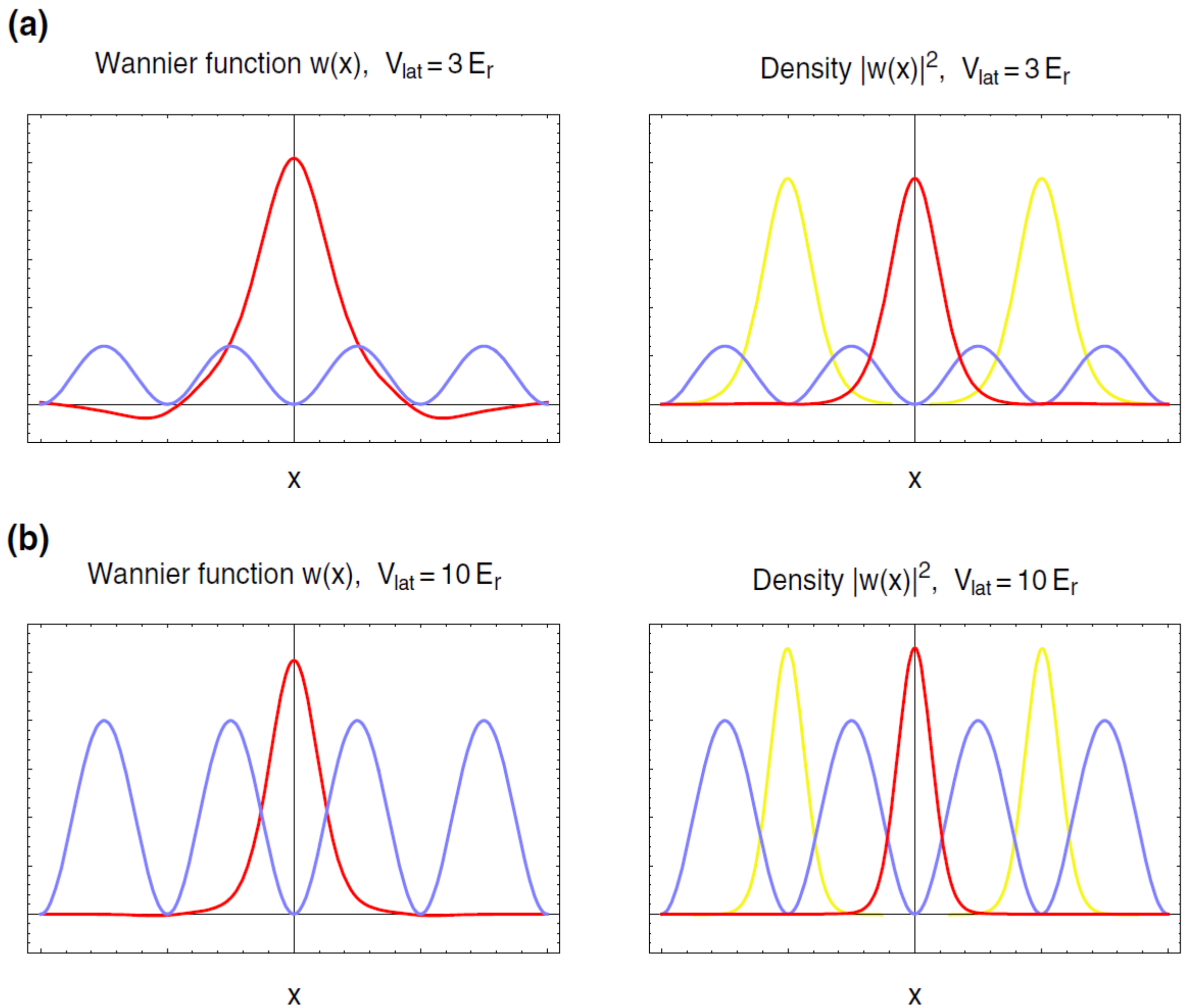


Band structure of an optical lattice



Wannier functions: well localized at lattice sites

$$w_n(x - x_i) = \mathcal{N}^{-1/2} \sum_q e^{-iqx_i/\hbar} \phi_q^{(n)}(x)$$



Bose-Hubbard model

Many-body Hamiltonian with contact interaction:

$$\hat{H} \;=\; \int d^3x\, \hat{\psi}^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) \right) \hat{\psi}(\mathbf{x}) + \frac{1}{2}\frac{4\pi a_s \hbar^2}{m} \int d^3x\, \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x}),$$

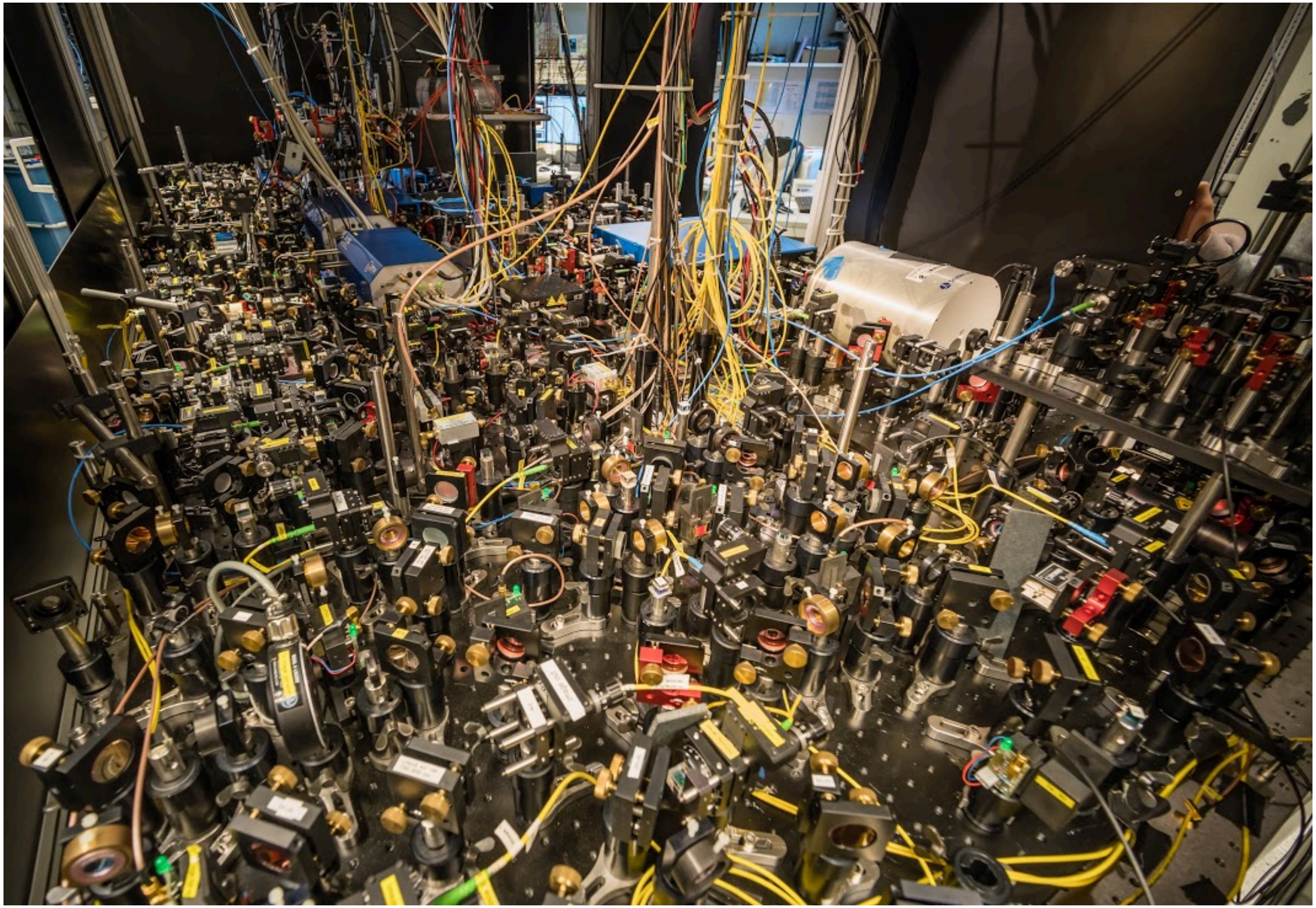
Modes: Wannier functions at the lowest band

$$\hat{\psi}(\mathbf{x}) = \sum_i \hat{a}_i w(\mathbf{x} - \mathbf{x}_i) \qquad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$$

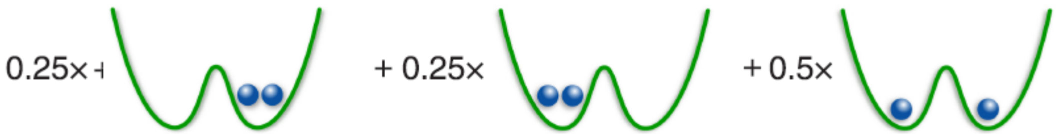
$$\hat{H} = -J \sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j + \sum_i (\epsilon_i - \mu)\, \hat{n}_i + \sum_i \frac{1}{2} U\, \hat{n}_i(\hat{n}_i - 1)$$

Competition between the tunnelling (spreads atoms)
and atom repulsion (localizes)

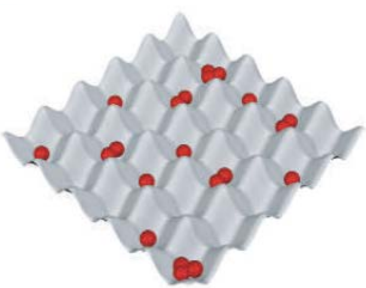
Real Bose-Hubbard model

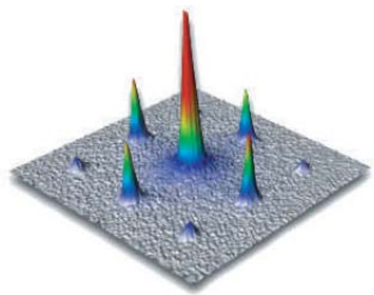


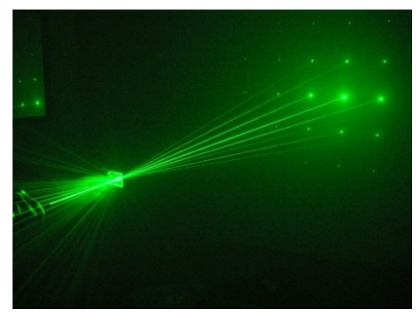
Superfluid state: each atom is delocalized over the whole lattice (fluctuations)



$$|\Psi_{SF}\rangle_{U/J\approx 0} \propto \left(\sum_{i=1}^{M} \hat{a}_i^{\dagger}\right)^{N} |0\rangle$$



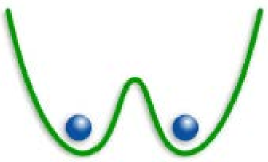




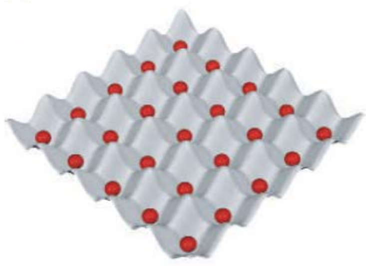
$$|\Psi_{\mathrm{Coh}}\rangle = \prod_{i=1}^{M} |\mathrm{Coh}\rangle_i$$

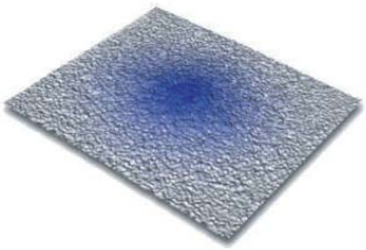
Approximation.
Describes: atom number fluctuations and interference.
Does not describe: correlations and entanglement between the sites.

Mott insulator state: atoms are localized (fluctuations suppressed)

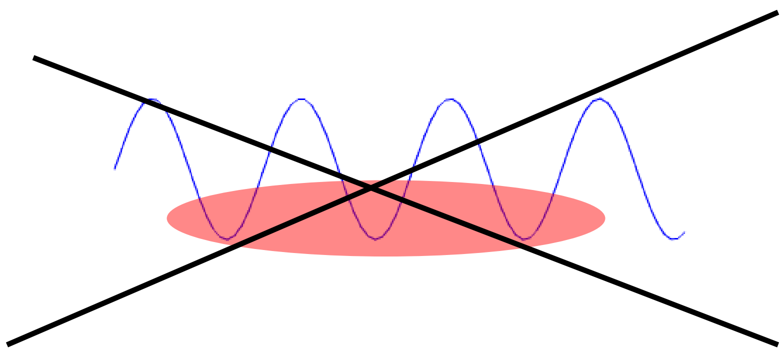


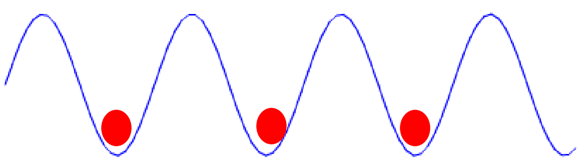
$$|\Psi_{MI}(n)\rangle_{J\approx 0} \propto \prod_{i=1}^{M} (\hat{a}_i^\dagger)^n |0\rangle$$

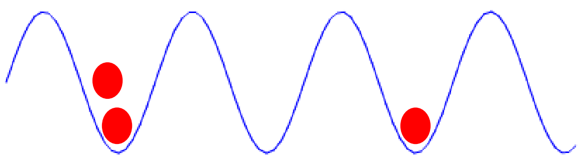




$$|1, 1, 1, \ldots 1, \ldots\rangle$$



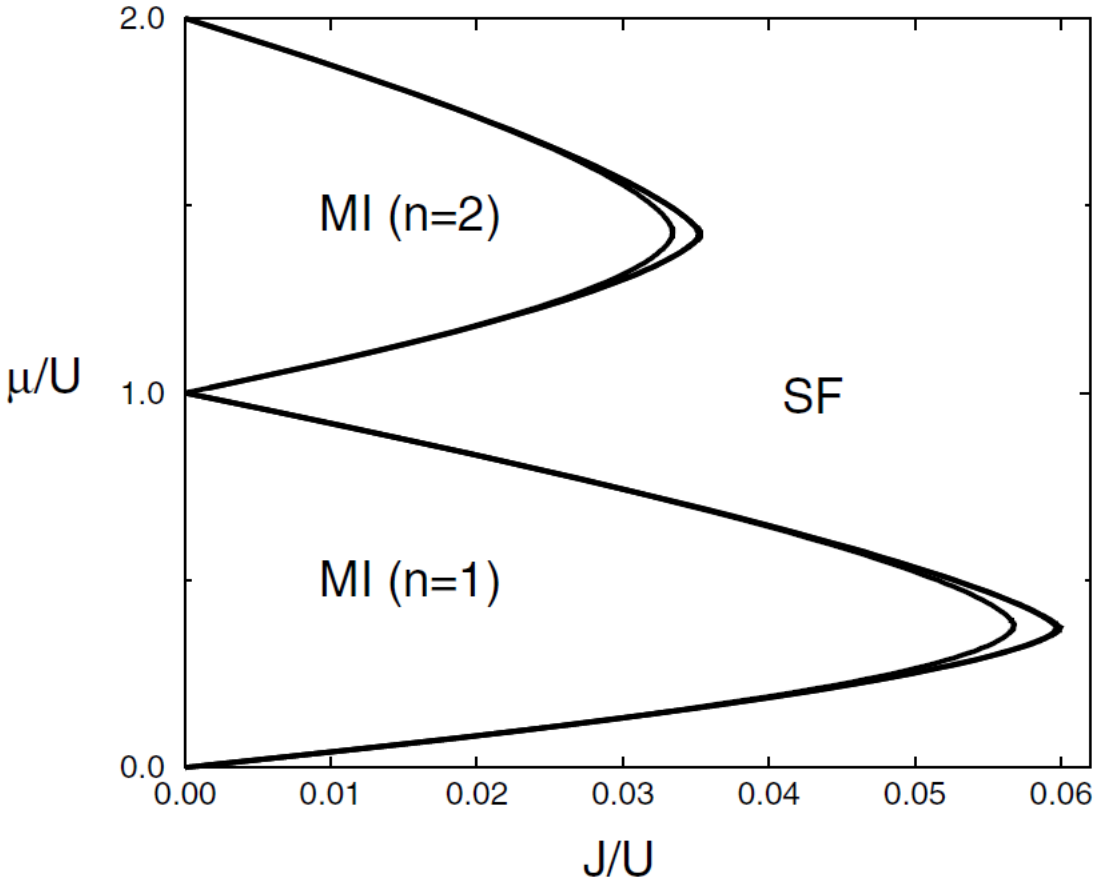




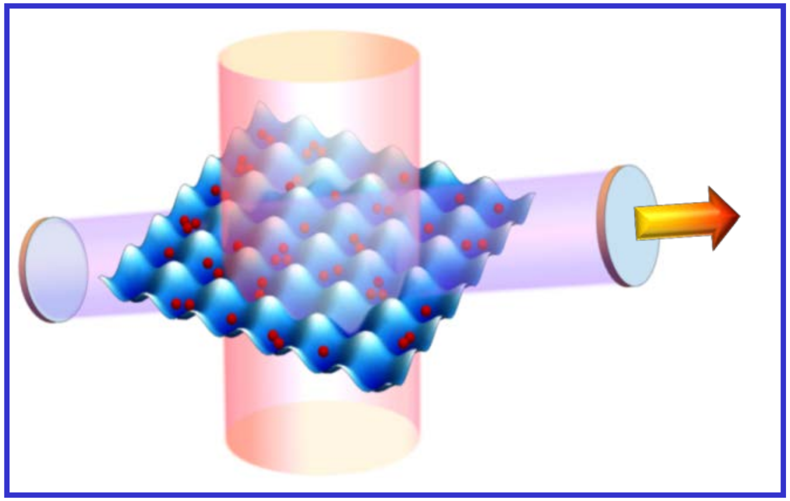
SF and MI: the same mean density, different number fluctuations

Quantum phase transition

$$\hat{H} = -J \sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j + \sum_i (\epsilon_i - \mu)\, \hat{n}_i + \sum_i \frac{1}{2} U\, \hat{n}_i(\hat{n}_i - 1)$$



Towards quantum optical lattices



Superfluid
Mott insulator

Density waves
Supersolid(-like)

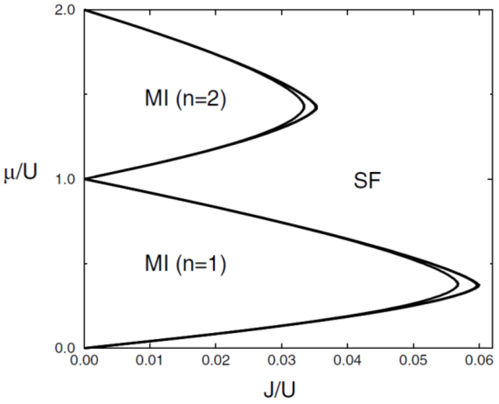
Dimers, trimers, …

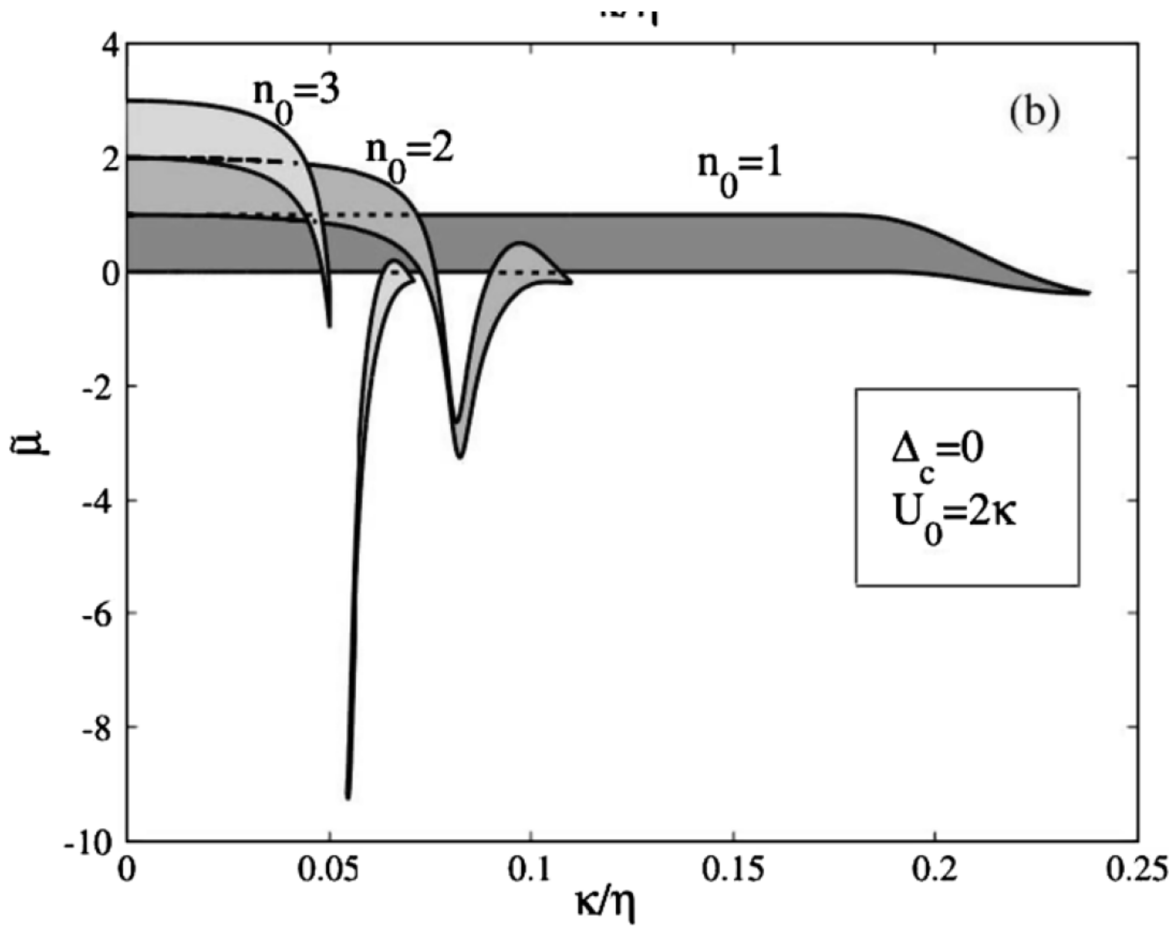
Quantum superpositions of phases

The potential is

(i) Dynamical
(ii) Quantum

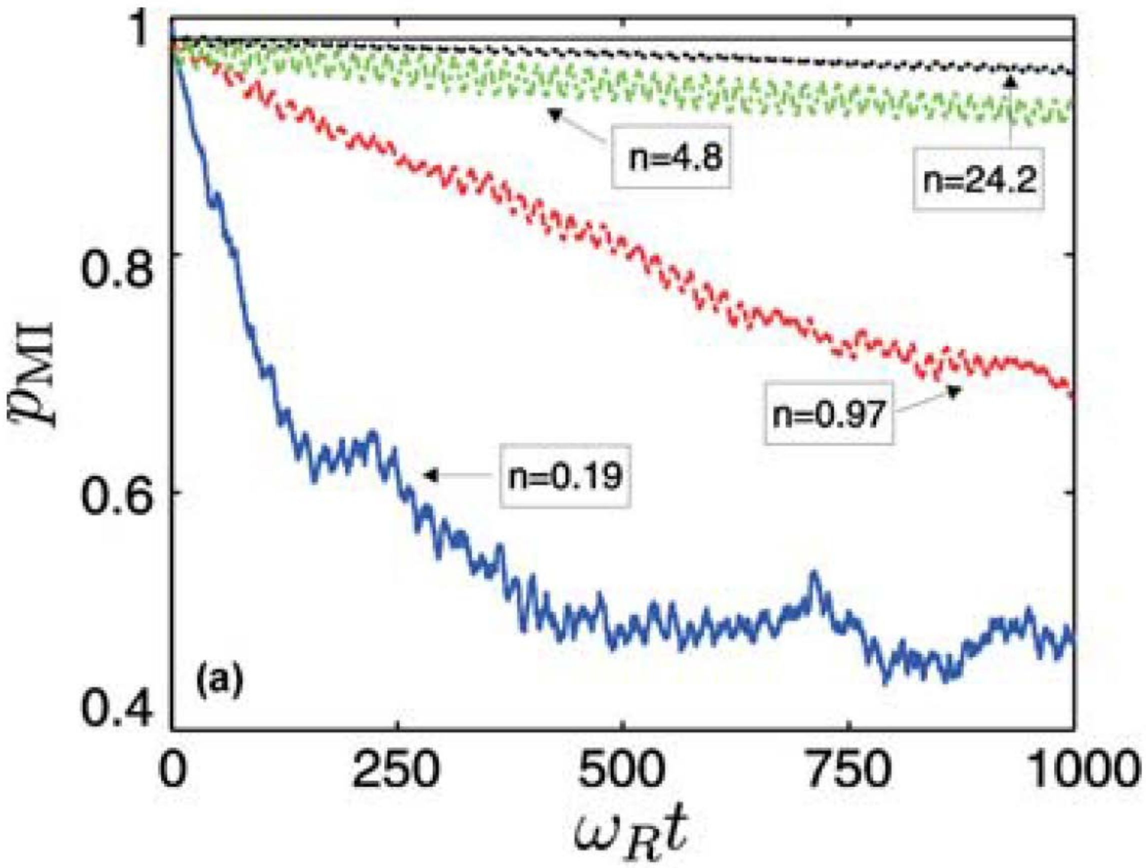
Overlap of the Mott insulator lobes (dynamical lattice)



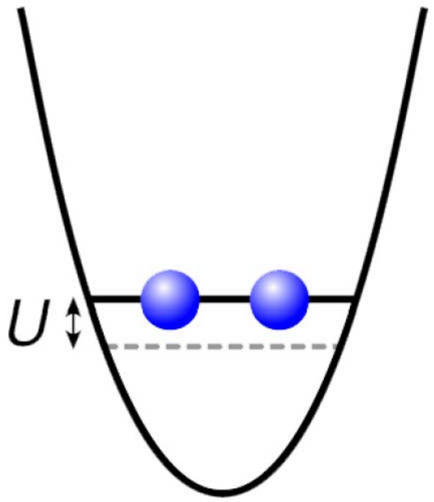


Leaving the MI ground state (quantum optical lattices)

average photon number in a cavity < 1



Collapse and revival of matter waves



$$\hat{H} = \frac{1}{2} U \, \hat{n}(\hat{n} - 1)$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \qquad \alpha = \sqrt{\bar{n}} \cdot e^{i\varphi}$$

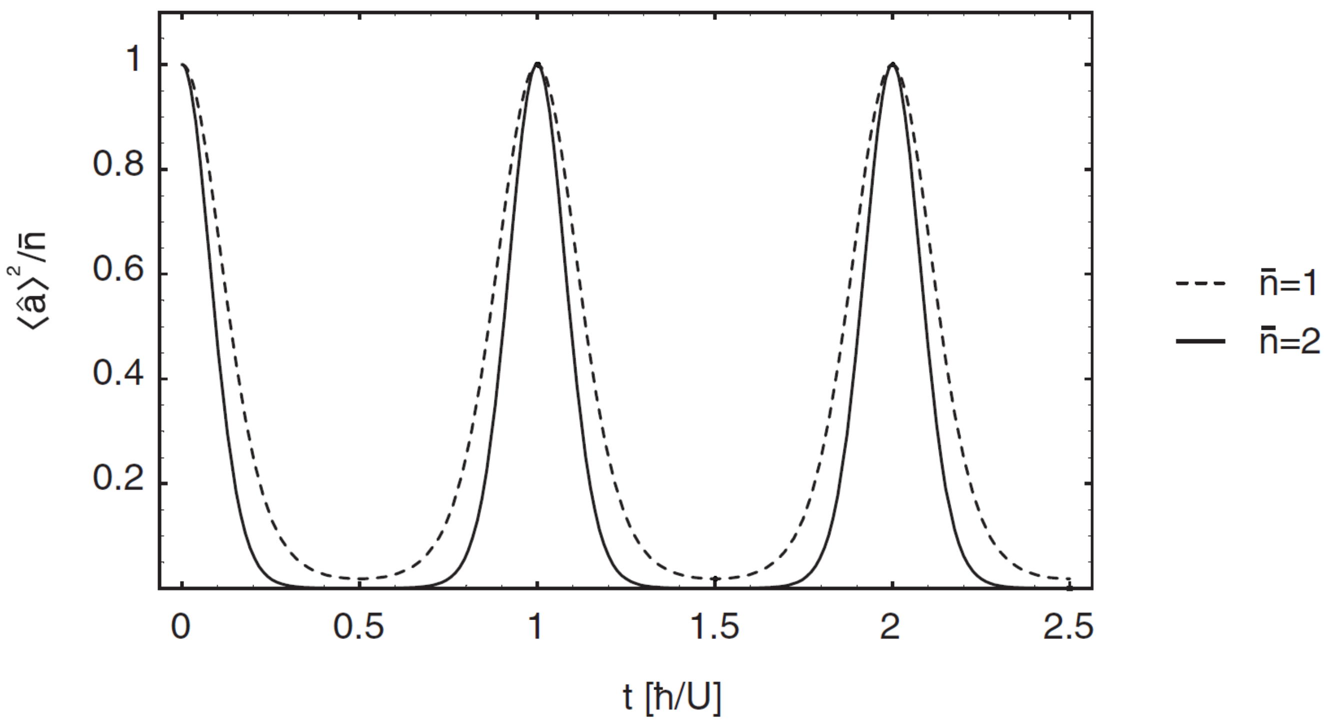
$$E_n = \frac{1}{2} U \, n(n-1)$$

$$|n\rangle(t) = |n\rangle \, e^{iE_n \, t/\hbar} = |n\rangle \, e^{i \, U \, n(n-1) \, t/2\hbar}$$

$$|\Phi(t)\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \cdot e^{i \, U \, n(n-1) \, t/2\hbar} \, |n\rangle$$

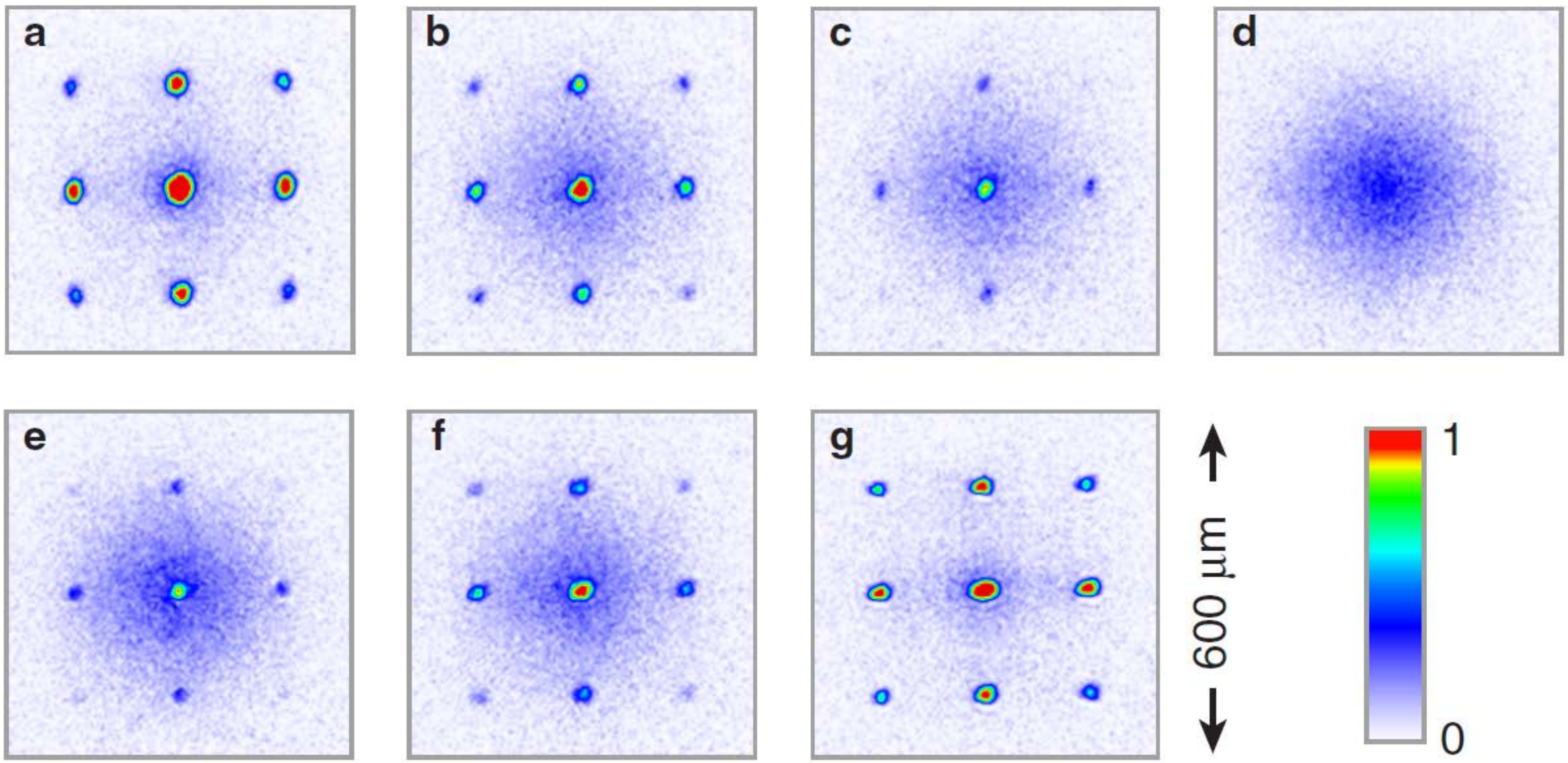
$$\psi(t) = \langle \Phi(t)|\hat{a}|\Phi(t)\rangle = \sqrt{\bar{n}} \sum_n \frac{\bar{n}^n e^{-\bar{n}}}{n!} \cdot e^{i(E_n - E_{n+1})t/\hbar}$$

$$= \sqrt{\bar{n}} \cdot \exp\left(\bar{n}(e^{-iUt/\hbar} - 1)\right)$$



Collapse $\psi(t) \simeq \sqrt{\bar{n}}\, e^{-i\bar{n}Ut/\hbar} \cdot e^{-\bar{n}U^2t^2/2\hbar^2}$ $t_c = \hbar/\sqrt{\bar{n}}U$

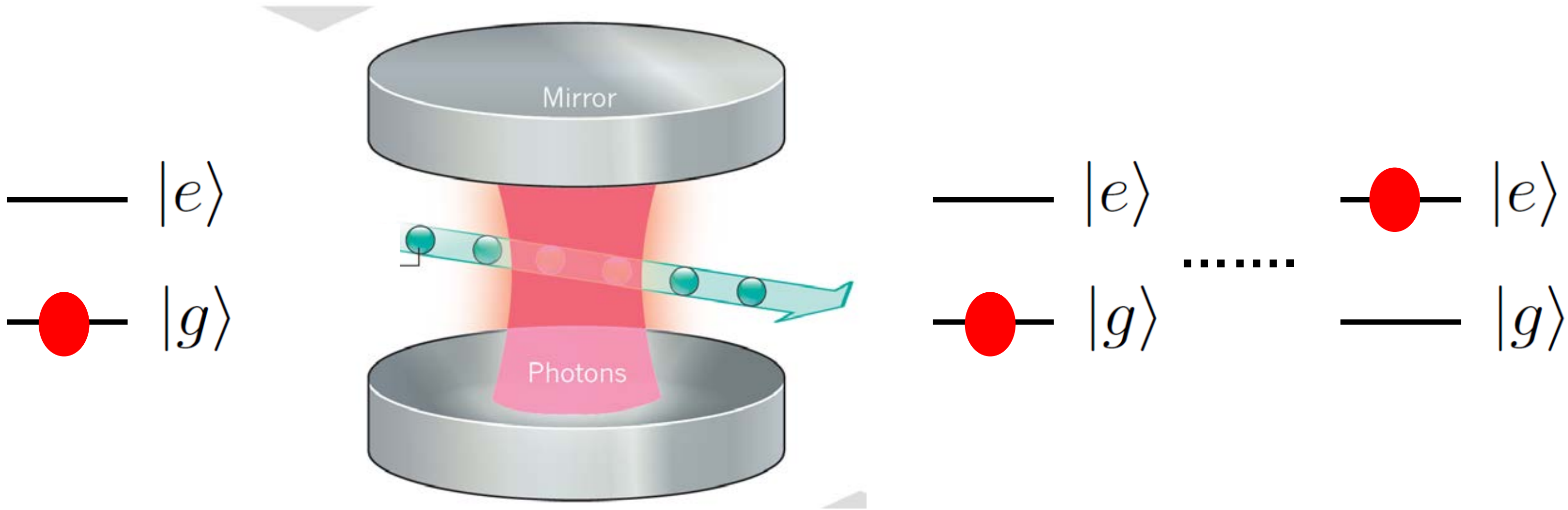
Revival $|\Phi(t = h/U)\rangle = |\Phi(t = 0)\rangle$ $t_{rev} = h/U$



a
b
c
d
e
f
g
600 µm
1
0

Quantum measurements

Probing the quantum state of photons in a cavity by sending atoms



S. Haroche

Hamiltonian in the interaction picture

$$H = \hbar g(a^\dagger|g\rangle\langle e| + a|e\rangle\langle g|)$$

Density matrix of the light field:

$$\rho(t+\tau) = \mathrm{Tr}_{\mathrm{atom}}\left(e^{-iH\tau/\hbar}\rho(t)|g\rangle\langle g|e^{iH\tau/\hbar}\right) =$$

$$\langle e|\rho_{\text{atom-light}}(t+\tau)|e\rangle + \langle g|\rho_{\text{atom-light}}(t+\tau)|g\rangle$$

Conditional density matrices (conditioned on the atomic state)

$$\rho_e(t+\tau) \qquad \rho_g(t+\tau)$$

$$\rho_e(t+\tau) \approx g^2\tau^2 a\rho(t)a^\dagger$$

$$\rho_g(t+\tau) \approx \rho(t) - \frac{1}{2}g^2\tau^2\left(a^\dagger a\rho(t) + \rho(t)a^\dagger a\right) \approx$$

$$e^{-R\tau a^\dagger a}\rho(t)e^{-R\tau a^\dagger a}$$

$$R = g^2\tau/2$$

$$\tau \to 0$$

Quantum jumps and evolution

$$\rho^{(n)} = e^{-S(t-t_n)} a e^{-S(t_n - t_{n-1})} \ldots a e^{-S(t_2-t_1)} a e^{-St_1} \rho(0) \times$$

$$e^{-St_1} a^\dagger e^{-S(t_2-t_1)} \ldots a^\dagger e^{-S(t_n-t_{n-1})} a^\dagger e^{-S(t-t_n)} / \mathrm{Tr}$$

$$S = Ra^\dagger a$$

After *n* photodetections

$$\rho^{(n)}(t) = \frac{e^{-Ra^\dagger at} a^n \rho(0) a^{\dagger n} e^{-Ra^\dagger at}}{\mathrm{Tr}(\rho(0) a^{\dagger n} e^{-2Ra^\dagger at} a^n)}$$

Stochastic evolution, Monte Carlo method

Assuming pure states:

$$|\psi^{(n)}(t+\delta t)\rangle = \frac{e^{-Ra^\dagger a\delta t}a^n|\psi(t)\rangle}{\sqrt{\langle\psi(t)|a^{\dagger n}e^{-2Ra^\dagger a\delta t}a^n|\psi(t)\rangle}}$$

No counts:

$$|\psi^{(0)}(t+\delta t)\rangle = \frac{e^{-Ra^\dagger a\delta t}|\psi(t)\rangle}{\sqrt{\langle\psi(t)|e^{-2Ra^\dagger a\delta t}|\psi(t)\rangle}} \approx \frac{1-Ra^\dagger a\delta t}{1-2R\langle a^\dagger a\rangle\delta t}|\psi(t)\rangle$$

Single count (quantum jump):

$$|\psi^{(1)}(t+\delta t)\rangle = \frac{e^{-Ra^\dagger a\delta t}a|\psi(t)\rangle}{\sqrt{\langle\psi(t)|a^\dagger e^{-2Ra^\dagger a\delta t}a|\psi(t)\rangle}} \approx \frac{a}{\langle a^\dagger a\rangle}|\psi(t)\rangle$$

Initial state

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$$

Evolution of the conditional state (unnormalized)

$$\frac{d}{dt}|\tilde{\psi}^{(0)}(t)\rangle = -\frac{i}{\hbar}(-i\hbar R a^\dagger a)|\tilde{\psi}^{(0)}(t)\rangle$$

Non-Hermitian Hamiltonian (with possible dynamics)

$$H_{n-H} = H - i\hbar R a^\dagger a$$

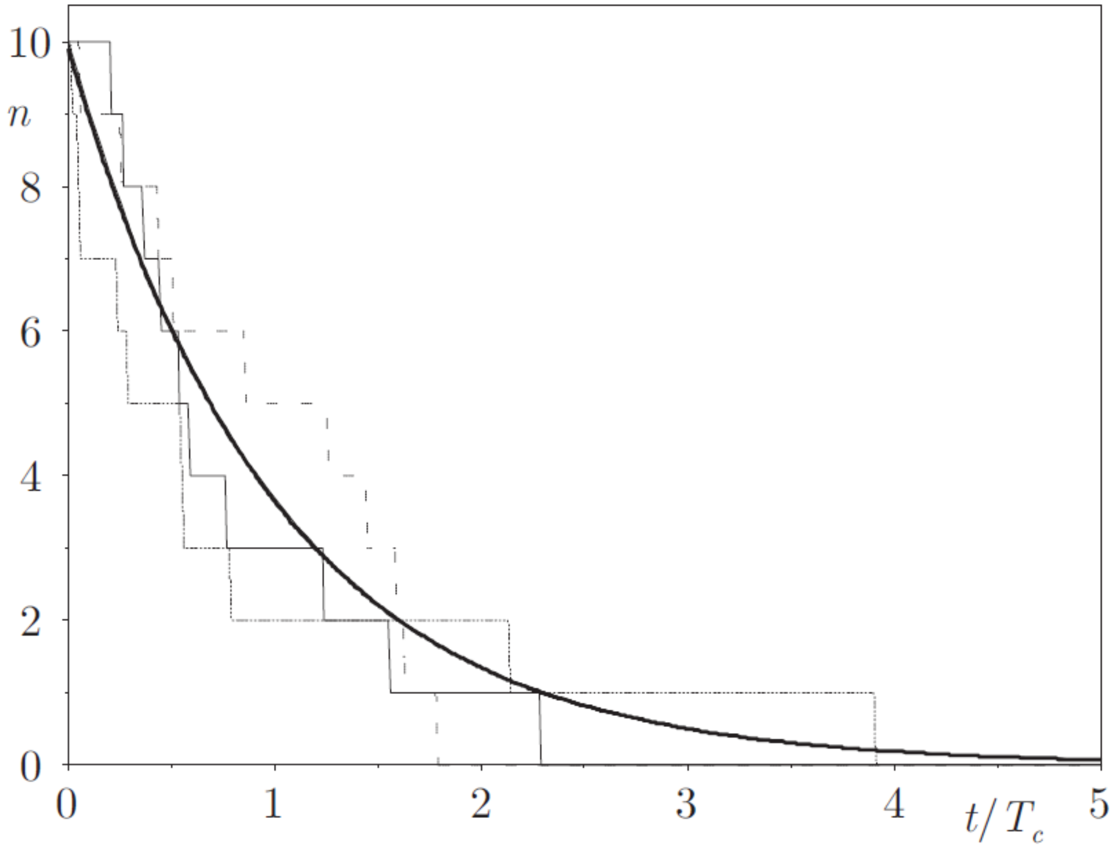
$$|\psi^{(0)}(t)\rangle = \frac{c_0(0)|0\rangle + c_1(0)e^{-Rt}|1\rangle}{\sqrt{|c_0(0)|^2 + |c_1(0)|^2 e^{-2Rt}}}$$

Quantum trajectory: intervals of non-Hermitian evolution and quantum jumps

Conditional and unconditional evolution

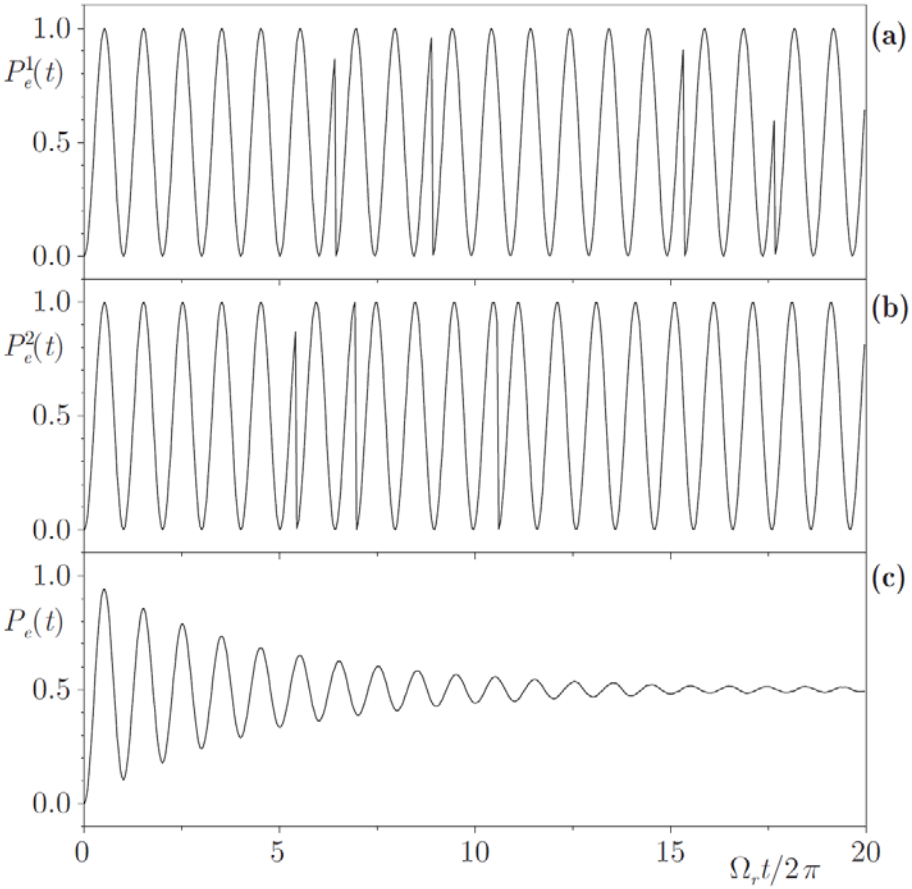
Decay of the initial Fock state of *n* photons:

Different stochastic quantum trajectories (conditional dynamics)
In average: the exponential decay (unconditional dynamics)



Spontaneous emission by a driven atom

Different stochastic quantum trajectories (conditional dynamics)
no steady state in a single trajectory
In average: the exponential decay (unconditional dynamics), steady state



Relation to the master equation (unconditional evolution):

$$\dot{\rho} = \frac{i}{\hbar}[H, \rho] - R(a^\dagger a\rho + \rho a^\dagger a) + 2Ra\rho a^\dagger$$

Conditional evolution: different unravellings are possible

Photodetection, quadrature (field amlitude and phase) measurements
Projection to different states

Dissipative phase transitions: steady state instead of ground states for quantum phase transitions

Quantum nondemolition measurements

Standard measurement introduces noise and affects dynamics and future measurements

Motion of a free particle:

$$H = \frac{p^2}{2m}$$

Uncertainty principle: measurement of the coordinate increases the uncertainty in the momentum

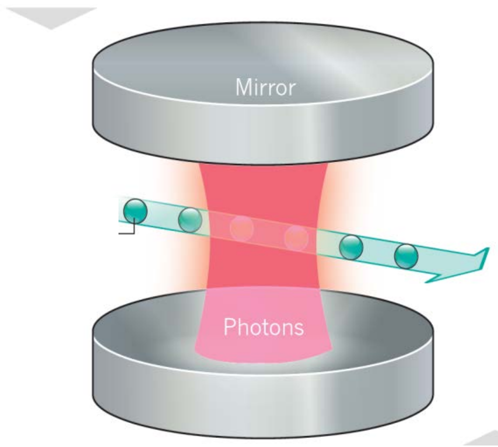
$$\Delta p \geq \frac{\hbar}{2\Delta x}$$

Uncertainty (noise) in the momentum affects the evolution of the coordinate

$$\dot{x} = \frac{1}{i\hbar}[x, H] = \frac{p}{m} \qquad x(t) = x(0) + \frac{p(0)t}{m}$$

Thus, the precision of the next measurement decreases.

QND: the conjugate variable is not in the Hamiltonian, thus while it is destroyed (becomes noisy), it does not affect the evolution



Dispersive off-resonant interaction

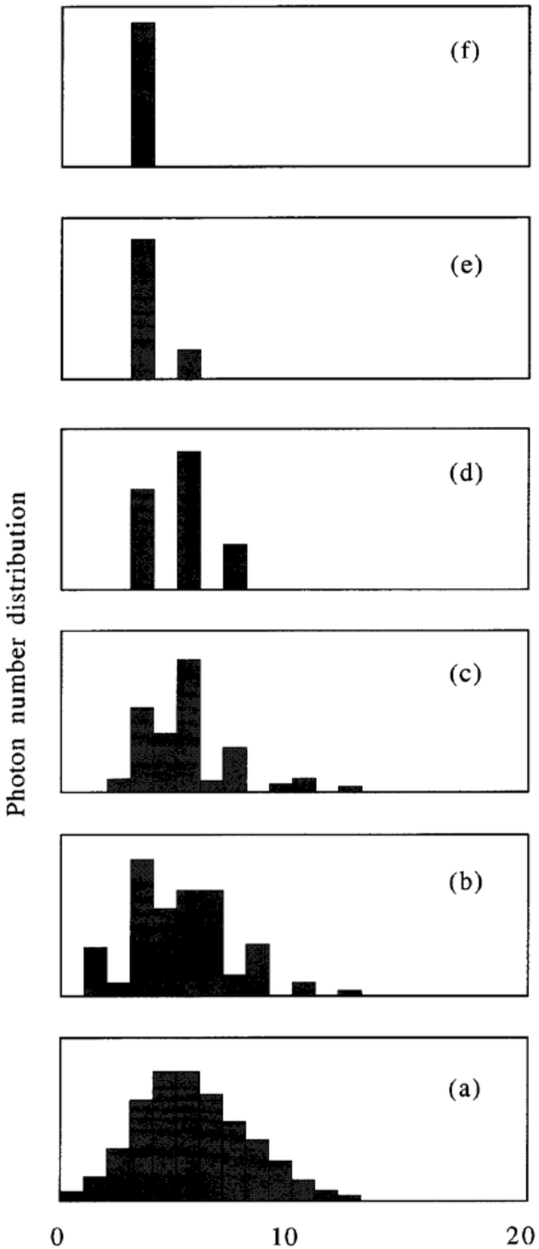
$$H = H_{\text{Signal}} + H_{\text{Probe}} + H_{\text{Interaction}}$$

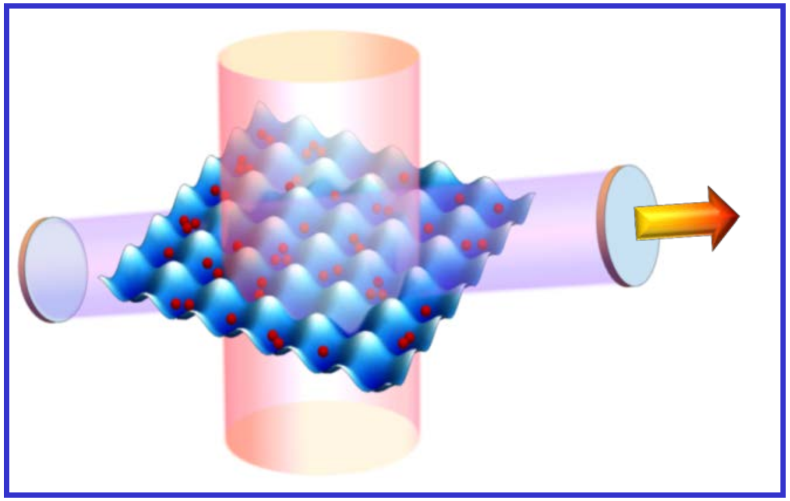
$$H_{\text{Signal}} = \hbar\omega a^\dagger a \qquad H_{\text{Probe}} = \frac{1}{2}\hbar\sigma_z$$

$$H_{\text{Interaction}} = \hbar\frac{g^2}{\Delta}a^\dagger a\sigma_+\sigma_-$$

QND measurement does change the quantum state, but does this in a minimally destructive way

“backaction evading measurements”





QND measurements of many-body variables

Conditional preparation of many-body states

Non-Hermitian many-body physics

Quantum feedback control beyond dissipative models