INSTITUTO DE FÍSICA
Universidade Federal Fluminense

Entangled structures in classical and quantum optics

Antonio Zelaquett Khoury

Outline

Lecture 1:

Optical vortices as entangled structures in classical and quantum optics

Lecture 2:

Quantum-like simulations and the role of quantum inequalities

Lecture 3:

Quantization of the electromagnetic field

Lecture 4:

Vector beam quantization and the unified framework

Outline

Lecture 4:

Vector beam quantization and the unified framework

Quantum fluctuations

Photon number (energy)

$$\left\langle \left(\Delta N_\mu\right)^2 \right\rangle = \langle \psi | \left(\Delta N_\mu\right)^2 | \psi \rangle = \langle \psi | N_\mu{}^2 | \psi \rangle - \langle \psi | N_\mu | \psi \rangle^2$$

Fock states: $\left\langle \left(\Delta N_\mu\right)^2 \right\rangle = 0$

Coherent states: $\left\langle \left(\Delta N_\mu\right)^2 \right\rangle = |\alpha|^2$

Poisson distribution:

$$|\alpha\rangle_\mu = e^{-|\alpha|^2/2} \sum_m \frac{\alpha^m}{\sqrt{m!}} |m\rangle_\mu$$

$$P(n) = \left| \langle n | \alpha \rangle_\mu \right|^2 = e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!}$$

$$\langle N_\mu \rangle = |\alpha|^2 \qquad \sqrt{\left\langle \left(\Delta N_\mu\right)^2 \right\rangle} = |\alpha|^2 = \langle N_\mu \rangle$$

Quadrature

$$\left\langle \left(\Delta X_\mu\right)^2 \right\rangle = \langle \psi | \left(\Delta X_\mu\right)^2 | \psi \rangle = \langle \psi | X_\mu{}^2 | \psi \rangle - \langle \psi | X_\mu | \psi \rangle^2$$

$$\left\langle \left(\Delta Y_\mu\right)^2 \right\rangle = \langle \psi | \left(\Delta Y_\mu\right)^2 | \psi \rangle = \langle \psi | Y_\mu{}^2 | \psi \rangle - \langle \psi | Y_\mu | \psi \rangle^2$$

Fock states:

$$\langle X_\mu \rangle = \langle Y_\mu \rangle = 0$$

$$\left\langle \left(\Delta X_\mu\right)^2 \right\rangle = \left\langle \left(\Delta Y_\mu\right)^2 \right\rangle = \hbar\left(n + \frac{1}{2}\right)$$

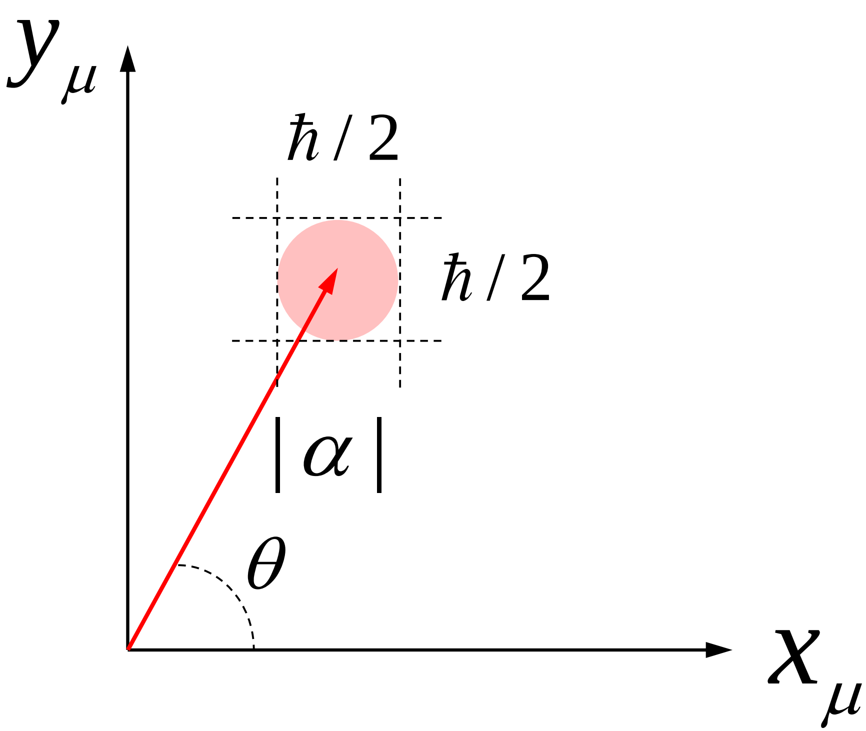
Coherent states:

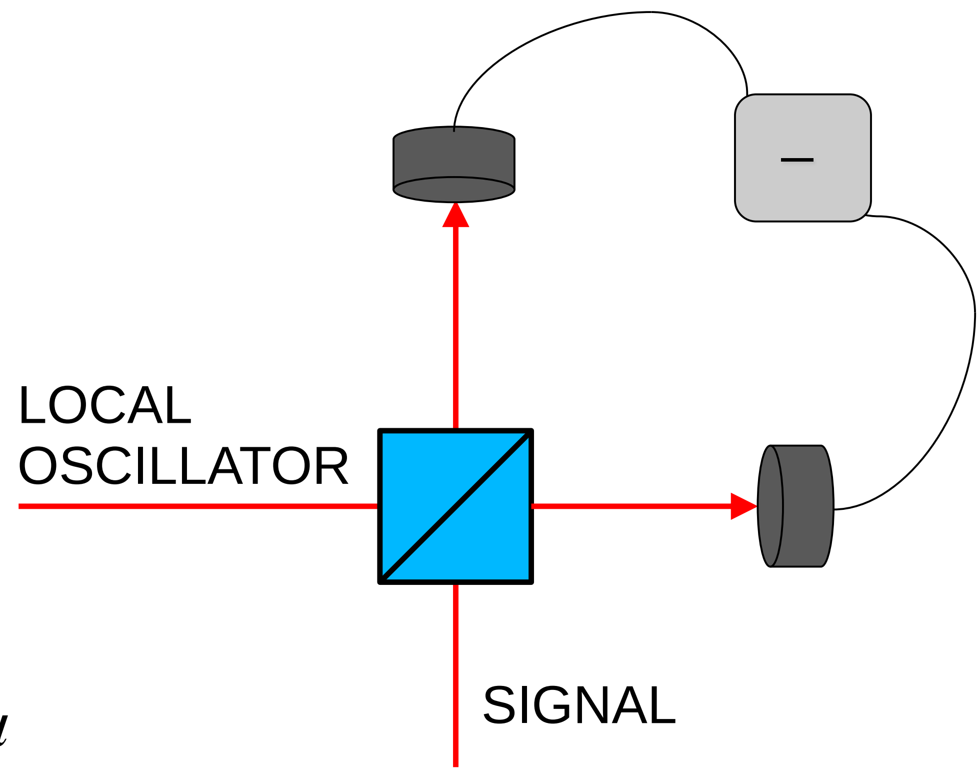
$$\langle X_\mu \rangle = \sqrt{\frac{\hbar}{2}}\left(\alpha + \alpha^*\right) \qquad \langle Y_\mu \rangle = -i\sqrt{\frac{\hbar}{2}}\left(\alpha - \alpha^*\right)$$

$$\left\langle \left(\Delta X_\mu\right)^2 \right\rangle = \left\langle \left(\Delta Y_\mu\right)^2 \right\rangle = \frac{\hbar}{2}$$

Phase space analysis

Quadrature measurement: homodyne detection

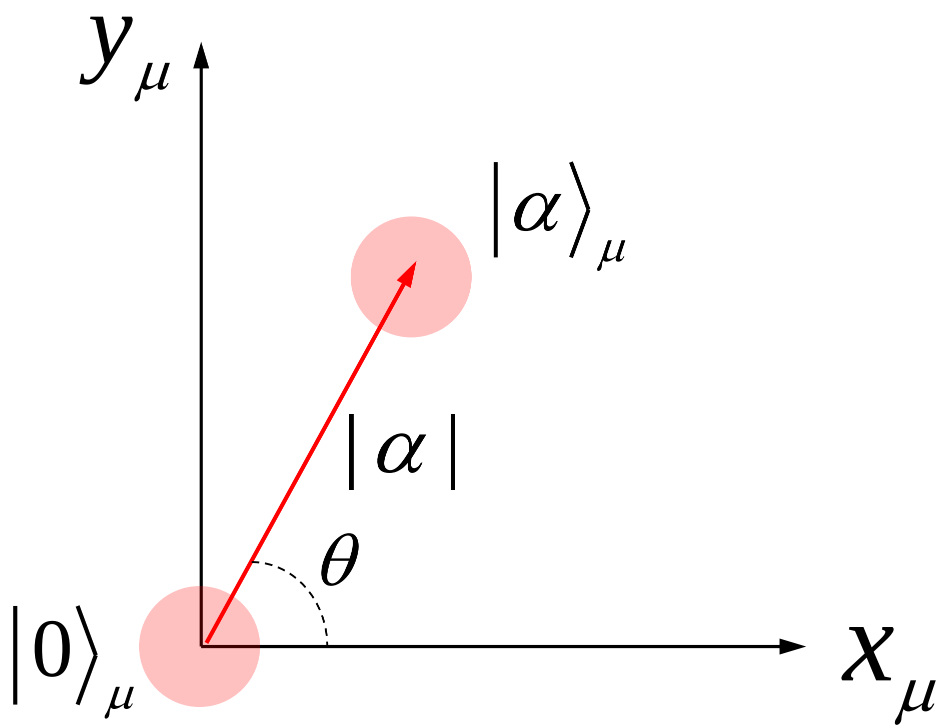




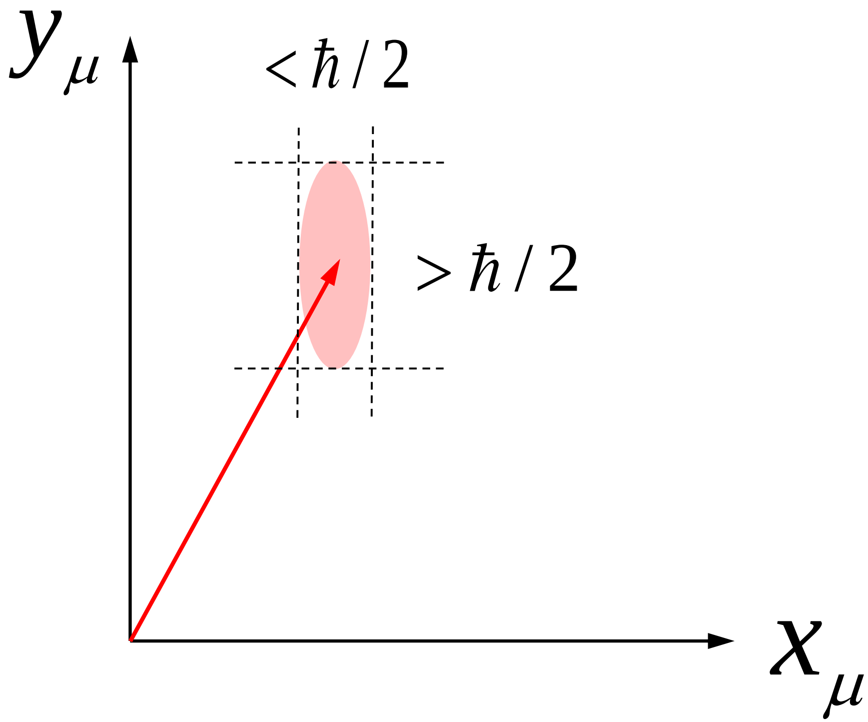
The Displacement Operator

$$D_\mu(\alpha) = \exp(\alpha^* a_\mu - \alpha a_\mu^\dagger) \quad \left(\alpha = |\alpha| e^{i\theta} \in \mathbb{C}\right)$$

$$D_\mu(\alpha)|0\rangle_\mu = |\alpha\rangle_\mu$$



Squeezed states



Minimum uncertainty

Squeezed states:

$$\sqrt{\left\langle \left(\Delta X_\mu\right)^2 \right\rangle \left\langle \left(\Delta Y_\mu\right)^2 \right\rangle} = \frac{\hbar}{2}$$

$$\left\langle \left(\Delta X_\mu\right)^2 \right\rangle \neq \left\langle \left(\Delta Y_\mu\right)^2 \right\rangle$$

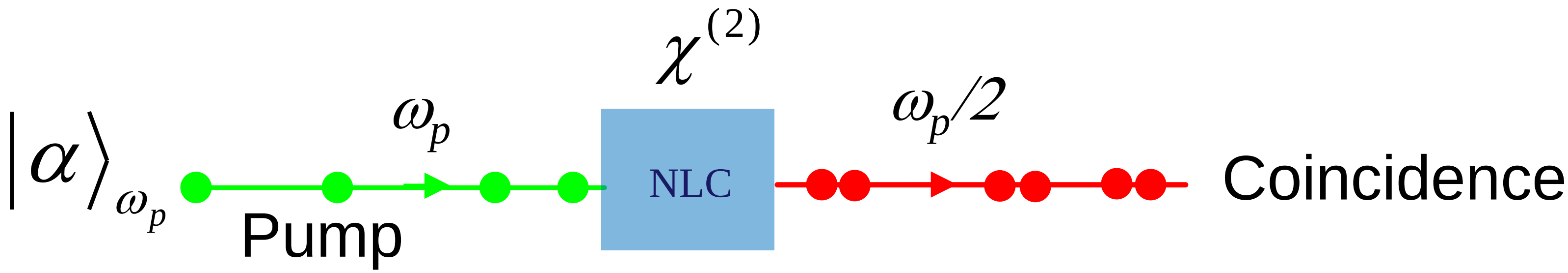
Squeeze operator

$$S_\mu(\xi) = \exp(\xi^* a_\mu^2 - \xi a_\mu^{2\dagger}) \quad \left(\xi = |\xi| e^{i\theta} \in \mathbb{C}\right)$$

$$S_\mu(\xi)\left|0\right\rangle_\mu = \left|\xi\right\rangle_\mu$$

Squeezed state generation

- Parametric down-conversion

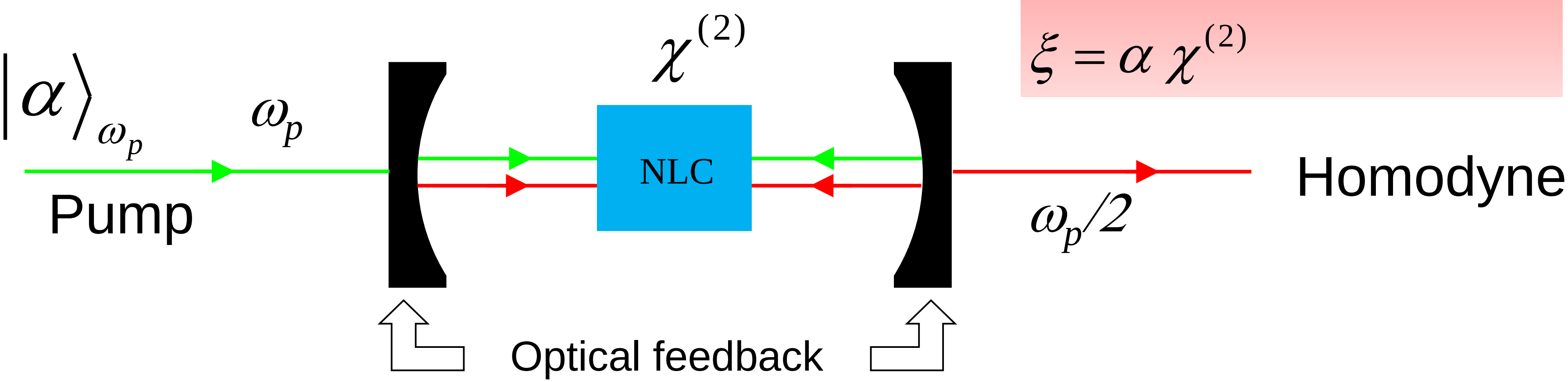


$$H_I = i\chi^{(2)}\left(a_{\omega_p} a^{2\dagger}_{\omega_p/2} - a^2_{\omega_p/2} a^\dagger_{\omega_p}\right) \approx i\chi^{(2)}\left(\alpha a^{2\dagger}_{\omega_p/2} - \alpha^* a^2_{\omega_p/2}\right)$$

- Optical parametric oscillator (OPO)

$$S_{\omega_p/2}(\xi)|0\rangle_{\omega_p/2} = |\xi\rangle_{\omega_p/2}$$

$$\xi = \alpha\,\chi^{(2)}$$



Multimode structure

Structure of the multimode vector space

Multimode vector space

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3 \cdots = \prod_{\mu}^{\otimes} \mathcal{E}_\mu$$

Multimode Fock states

$$\left| n_1, n_2, n_3, \ldots \right\rangle = \left| n_1 \right\rangle_1 \otimes \left| n_2 \right\rangle_2 \otimes \left| n_3 \right\rangle_3 \otimes \cdots \quad \Rightarrow \quad \left| \{ n_\mu \} \right\rangle = \prod_{\mu}^{\otimes} \left| n_\mu \right\rangle_\mu$$

Operator extension

$$O_\nu \to O_\nu \otimes \prod_{\mu \neq \nu}^{\otimes} \mathbf{1}_\mu$$

$$a_\nu^\dagger a_\nu \left| \{ n_\mu \} \right\rangle = n_\nu \left| \{ n_\mu \} \right\rangle$$

Quantized fields

Electromagnetic fields

$$\hat{\mathbf{A}}(\mathbf{r},t)=\sum_{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_\mu}}\left[a_\mu(t)\mathbf{u}_\mu(\mathbf{r})+a_\mu^\dagger(t)\mathbf{u}_\mu^*(\mathbf{r})\right]$$

$$\hat{\mathbf{E}}(\mathbf{r},t)=i\sum_{\mu}\sqrt{\frac{\hbar\omega_\mu}{2\varepsilon_0 V}}\left[a_\mu(t)\mathbf{u}_\mu(\mathbf{r})-a_\mu^\dagger(t)\mathbf{u}_\mu^*(\mathbf{r})\right]$$

$$\hat{\mathbf{B}}(\mathbf{r},t)=\sum_{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_\mu}}\left[a_\mu(t)\nabla\times\mathbf{u}_\mu+a_\mu^\dagger(t)\nabla\times\mathbf{u}_\mu^*\right]$$

$$H=\sum_{\mu}\hbar\omega_\mu\left(a_\mu^\dagger a_\mu+\frac{1}{2}\right)$$

The vacuum state

$$|\mathrm{vac}\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes \cdots = \prod_{\mu}^{\otimes} |0\rangle_\mu$$

Zero point energy

$$\langle \mathrm{vac} | H | \mathrm{vac} \rangle = \sum_{\mu} \frac{\hbar\omega_\mu}{2}$$

$$\langle \mathrm{vac} | \hat{\mathbf{E}} | \mathrm{vac} \rangle = 0$$

$$\langle \mathrm{vac} | \Delta\hat{\mathbf{E}}^2 | \mathrm{vac} \rangle \neq 0$$

→

- Quantum fluctuations
- Spontaneous emission
- Casimir force

Multimode coherent states

Tensor product of coherent states

$$\left|\alpha_1\right\rangle_1 \otimes \left|\alpha_2\right\rangle_2 \otimes \left|\alpha_3\right\rangle_3 \otimes \cdots \equiv \prod_{\mu}^{\otimes} \left|\alpha_\mu\right\rangle_\mu \equiv \left|\alpha_1, \alpha_2, \alpha_3, \ldots\right\rangle \equiv \left|\{\alpha_\mu\}\right\rangle$$

$$a_\nu \left|\{\alpha_\mu\}\right\rangle = \alpha_\nu \left|\{\alpha_\mu\}\right\rangle$$

$$\left\langle\{\alpha_\mu\}\right|\hat{\mathbf{E}}(\mathbf{r},t)\left|\{\alpha_\mu\}\right\rangle = i\sum_\mu \sqrt{\frac{\hbar\omega_\mu}{2\varepsilon_0 V}}\left[\alpha_\mu(t)\mathbf{u}_\mu(\mathbf{r}) - \alpha_\mu^*(t)\mathbf{u}_\mu^*(\mathbf{r})\right]$$

$$D(\alpha_1,\ldots,\alpha_n) = \exp(\alpha_1^* a_1 - \alpha_1 a_1^\dagger + \ldots + \alpha_n^* a_n - \alpha_n a_n^\dagger) = D(\alpha_1)\cdots D(\alpha_n)$$

$$D(\alpha_1,\ldots,\alpha_n)\left|\mathrm{vac}\right\rangle = \left|\alpha\right\rangle_1 \otimes \cdots \otimes \left|\alpha\right\rangle_n$$

Multimode displacement

Simple example

Example: plane wave, *x*-polarized, propagating along *z*

$$\mathbf{u}_{x,0,0,k}(\mathbf{r}) = e^{ikz}\hat{\mathbf{x}}$$

$$\alpha_{x,0,0,k} = \alpha$$

$$\alpha_\mu = 0 \quad \text{for} \quad \mu \neq (x,0,0,k)$$

$$\left\langle \{\alpha_\mu\} \middle| \hat{\mathbf{E}}(\mathbf{r},t) \middle| \{\alpha_\mu\} \right\rangle = \sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}}\,|\alpha(0)|\sin(kz-\omega t+\theta)\,\hat{\mathbf{x}}$$

Classical-like behaviour!

Mode transformations

Alternative modes

$$\{\mathbf{u}_\mu(\mathbf{r})\} \Leftrightarrow \{\mathbf{v}_\mu(\mathbf{r})\}$$

$$\int_V \mathbf{u}_\mu^*(\mathbf{r}) \cdot \mathbf{u}_\nu(\mathbf{r})\, d^3\mathbf{r} = \delta_{\mu\nu}$$

$$\int_V \mathbf{v}_\mu^*(\mathbf{r}) \cdot \mathbf{v}_\nu(\mathbf{r})\, d^3\mathbf{r} = \delta_{\mu\nu}$$

Mode transformation

$U \longrightarrow$ unitary

$$\mathbf{v}_\mu(\mathbf{r}) = \sum_\nu U_{\mu\nu} \mathbf{u}_\nu(\mathbf{r})$$

$$\mathbf{u}_\mu(\mathbf{r}) = \sum_\nu U_{\mu\nu}^\dagger \mathbf{v}_\nu(\mathbf{r})$$

OBS: $\omega_\mu = \omega_\nu$

Alternative modes

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_\mu}}a_\mu(t)\mathbf{u}_\mu(\mathbf{r}) + \text{h.c.}$$

$$= \sum_{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_\mu}}\, a_\mu(t)\sum_{\nu}\left(U^\dagger\right)_{\mu\nu}\mathbf{v}_\nu(\mathbf{r}) + \text{h.c.}$$

$$= \sum_{\nu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_\nu}}\left[\sum_{\mu}\left(U^\dagger\right)_{\mu\nu} a_\mu(t)\right]\mathbf{v}_\nu(\mathbf{r}) + \text{h.c.}$$

$$= \sum_{\nu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_\nu}}\left[\sum_{\mu}U^*_{\nu\mu}a_\mu(t)\right]\mathbf{v}_\nu(\mathbf{r}) + \text{h.c.}$$

$$= \sum_{\nu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_\nu}}\, b_\nu(t)\,\mathbf{v}_\nu(\mathbf{r}) + \text{h.c.}$$

$$b_\nu(t) = \sum_{\mu}U^*_{\nu\mu}a_\mu(t) \quad \Rightarrow \quad \left[b^\dagger_\alpha, b_\beta\right] = \delta_{\alpha\beta}$$

Show this

Alternative modes

$$b_\nu = \sum_\mu U^*_{\nu\mu} a_\mu \Rightarrow b^\dagger_\nu = \sum_\mu U_{\nu\mu} a^\dagger_\mu$$

$$\sum_\nu b^\dagger_\nu b_\nu = \sum_\nu \left(\sum_\mu U_{\nu\mu} a^\dagger_\mu \right) \left(\sum_{\mu'} U^*_{\nu\mu'} a_{\mu'} \right)$$

$$= \sum_{\mu\mu'} \underbrace{\left(\sum_\nu U^\dagger_{\mu'\nu} U_{\nu\mu} \right)}_{U^\dagger U = I \rightarrow \delta_{\mu\mu'}} a^\dagger_\mu a_{\mu'} = \sum_\mu a^\dagger_\mu a_\mu$$

$U \rightarrow$ unitary

$$H = \sum_\mu \hbar\omega_\mu \left(a^\dagger_\mu a_\mu + \frac{1}{2} \right) = \sum_\mu \hbar\omega_\mu \left(b^\dagger_\mu b_\mu + \frac{1}{2} \right)$$

Fock states transformation

$$\{\mathbf{u}_\mu(\mathbf{r})\} \Leftrightarrow \{\mathbf{v}_\nu(\mathbf{r})\}$$

$$\left|\{n_\mu\}\right\rangle \Leftrightarrow \left|\{n_\nu\}\right\rangle$$

$$b_\nu = \sum_\mu U^*_{\nu\mu} a_\mu \Rightarrow b^\dagger_\nu = \sum_\mu U_{\nu\mu} a^\dagger_\mu$$

$$\left|\{0_\nu\}\right\rangle = \left|\{0_\mu\}\right\rangle = \left|\mathrm{vac}\right\rangle$$

$$\left|n_\nu, \{0_{\nu'\neq\nu}\}\right\rangle = \frac{\left(b^\dagger_\nu\right)^n}{\sqrt{n!}}\left|\mathrm{vac}\right\rangle = \frac{1}{\sqrt{n!}}\left(\sum_\mu U_{\nu\mu} a^\dagger_\mu\right)^n \left|\mathrm{vac}\right\rangle$$

$$\frac{1}{\sqrt{n!}}\left(\sum_\mu U_{\nu\mu} a^\dagger_\mu\right)^n \left|\mathrm{vac}\right\rangle = \frac{n!}{\sqrt{n!}} \sum_{\{m_\mu\}} \prod_\mu \frac{\left(U_{\nu\mu} a^\dagger_\mu\right)^{m_\mu}}{m_\mu!}\left|\mathrm{vac}\right\rangle = \sqrt{n!}\sum_{\{m_\mu\}} \prod_\mu^{\otimes} \frac{\left(U_{\nu\mu}\right)^{m_\mu}}{\sqrt{m_\mu!}}\left|m_\mu\right\rangle_\mu$$

where: $\sum_\mu m_\mu = n$

Coherent states transformation

$$\left\{\mathbf{u}_\mu(\mathbf{r})\right\} \Leftrightarrow \left\{\mathbf{v}_\nu(\mathbf{r})\right\}$$

$$\left|\left\{\alpha_\mu\right\}\right\rangle \Leftrightarrow \left|\left\{\beta_\nu\right\}\right\rangle$$

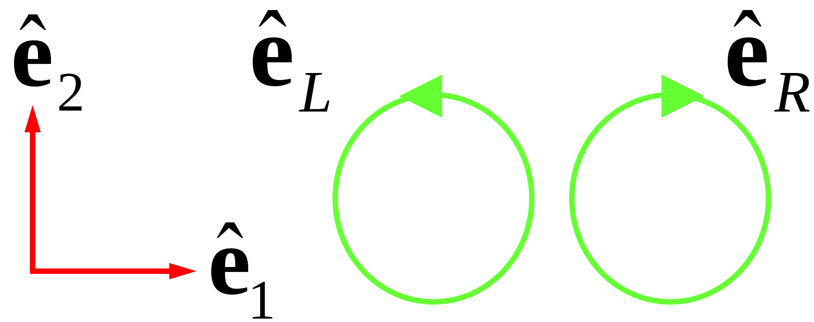
$$b_\nu = \sum_\mu U^*_{\nu\mu} a_\mu \Rightarrow b^\dagger_\nu = \sum_\mu U_{\nu\mu} a^\dagger_\mu$$

$$\left|\beta_\nu, \left\{0_{\nu'\neq\nu}\right\}\right\rangle = e^{\beta_\nu b^\dagger_\nu - \beta^*_\nu b_\nu} \left|\mathrm{vac}\right\rangle = \underbrace{\exp\left(\sum_\mu \beta_\nu U_{\nu\mu} a^\dagger_\mu - \beta^*_\nu U^*_{\nu\mu} a_\mu\right)}_{\prod_\mu^{\otimes} D_\mu(\alpha_\mu)} \left|\mathrm{vac}\right\rangle$$

$$\left|\beta_\nu, \left\{0_{\nu'\neq\nu}\right\}\right\rangle = \left|\left\{\alpha_\mu\right\}\right\rangle \text{ where } \alpha_\mu = \beta_\nu U_{\nu\mu}$$

Simple examples

Circular polarization modes



$$\hat{\mathbf{e}}_L = \frac{\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2}{\sqrt{2}}$$

$$\hat{\mathbf{e}}_R = \frac{\hat{\mathbf{e}}_1 - i\hat{\mathbf{e}}_2}{\sqrt{2}}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

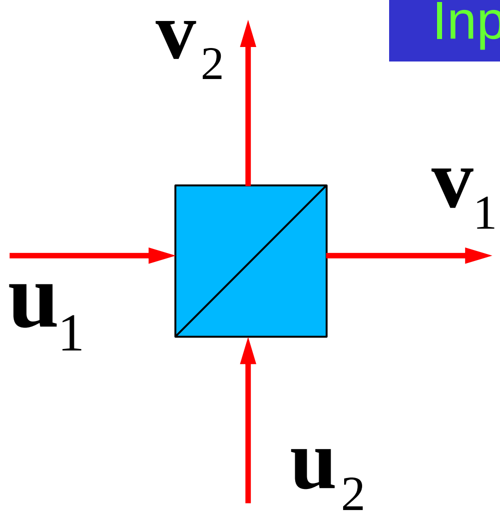
$$b_L = \frac{a_1 - ia_2}{\sqrt{2}} \qquad b_L^\dagger = \frac{a_1^\dagger + ia_2^\dagger}{\sqrt{2}}$$

$$b_R = \frac{a_1 + ia_2}{\sqrt{2}} \qquad b_R^\dagger = \frac{a_1^\dagger - ia_2^\dagger}{\sqrt{2}}$$

$$\Rightarrow \quad b_R\hat{\mathbf{e}}_R + b_L\hat{\mathbf{e}}_L = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2$$

Simple examples

Input-output modes of a beam splitter



$$\mathbf{v}_1 = \frac{\mathbf{u}_1 + \mathbf{u}_2}{\sqrt{2}}$$

$$\mathbf{v}_2 = \frac{-\mathbf{u}_1 + \mathbf{u}_2}{\sqrt{2}}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$b_1 = \frac{a_1 + a_2}{\sqrt{2}} \qquad b_1^\dagger = \frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}}$$

$$b_2 = \frac{a_1 - a_2}{\sqrt{2}} \qquad b_2^\dagger = \frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}$$

$$\Rightarrow \quad b_1\mathbf{v}_1 + b_2\mathbf{v}_2 = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$$

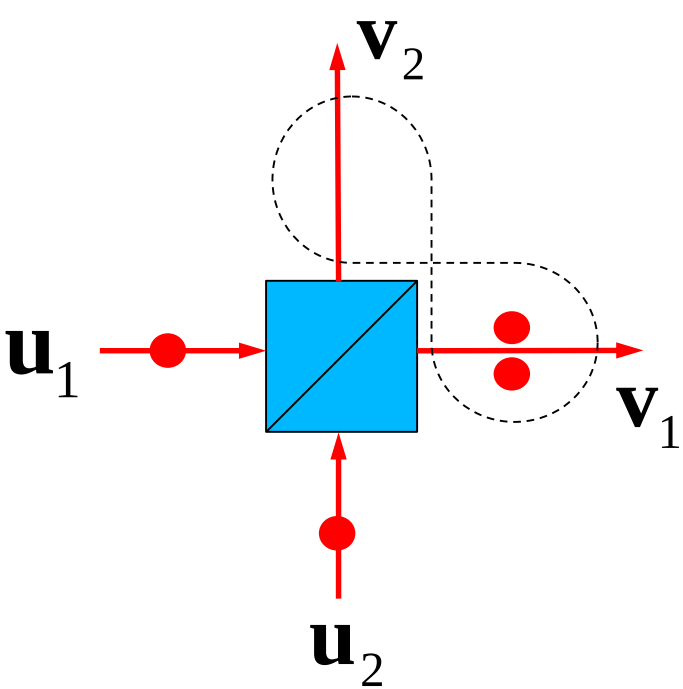
$$|n\rangle_{\mathbf{v}_1} |N-n\rangle_{\mathbf{v}_2} = \frac{\left(b_{\mathbf{v}_1}^\dagger\right)^n \left(b_{\mathbf{v}_2}^\dagger\right)^{N-n}}{\sqrt{n!(N-n)!}} |\mathrm{vac}\rangle = \frac{1}{\sqrt{n!(N-n)!}} \left(\frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}}\right)^n \left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}\right)^{N-n} |\mathrm{vac}\rangle$$

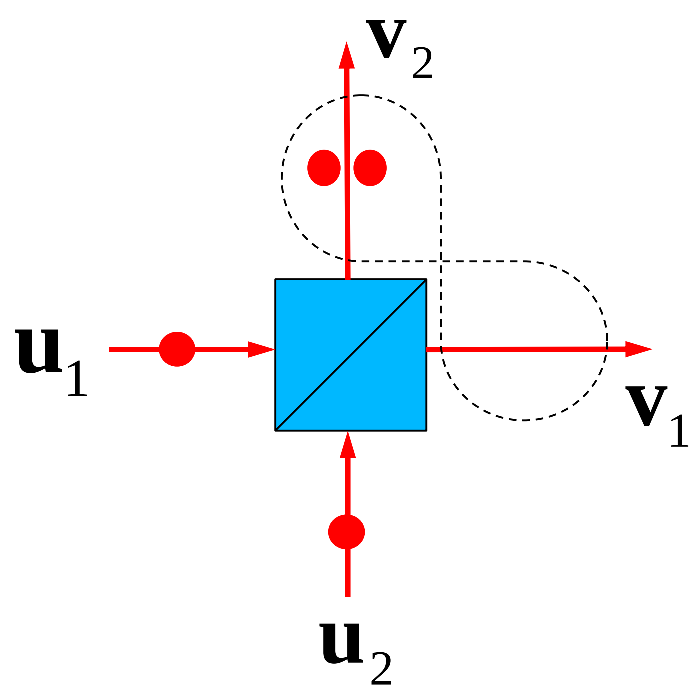
Hong-Ou-Mandel Effect

Input-output of a photon pair on a beam splitter

$$|1\rangle_{\mathbf{u}_1}|1\rangle_{\mathbf{u}_2} = a^\dagger_{\mathbf{u}_1} a^\dagger_{\mathbf{u}_2}|\text{vac}\rangle = \left(\frac{b^\dagger_{\mathbf{v}_1} + b^\dagger_{\mathbf{v}_2}}{\sqrt{2}}\right)\left(\frac{b^\dagger_{\mathbf{v}_1} - b^\dagger_{\mathbf{v}_2}}{\sqrt{2}}\right)|\text{vac}\rangle = \left(\frac{\left(b^\dagger_{\mathbf{v}_1}\right)^2 - \left(b^\dagger_{\mathbf{v}_2}\right)^2}{2}\right)|\text{vac}\rangle$$

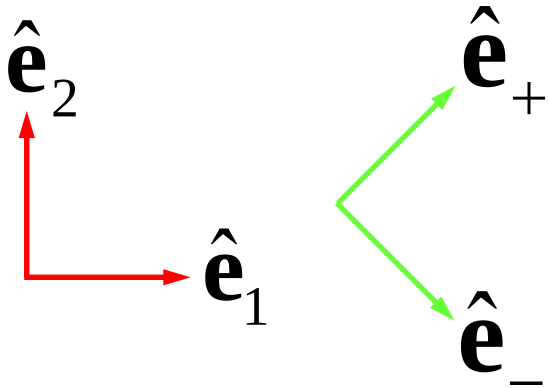
$$|1\rangle_{\mathbf{u}_1}|1\rangle_{\mathbf{u}_2} = \frac{|2\rangle_{\mathbf{v}_1}|0\rangle_{\mathbf{v}_2} - |0\rangle_{\mathbf{v}_1}|2\rangle_{\mathbf{v}_2}}{\sqrt{2}}$$





Simple examples

Linear polarization modes



$$\hat{\mathbf{e}}_+ = \frac{\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{2}}$$

$$\hat{\mathbf{e}}_- = \frac{\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2}{\sqrt{2}}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$b_+ = \frac{a_1 + a_2}{\sqrt{2}} \qquad b_L^\dagger = \frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}}$$

$$b_- = \frac{a_1 - a_2}{\sqrt{2}} \qquad b_R^\dagger = \frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}$$

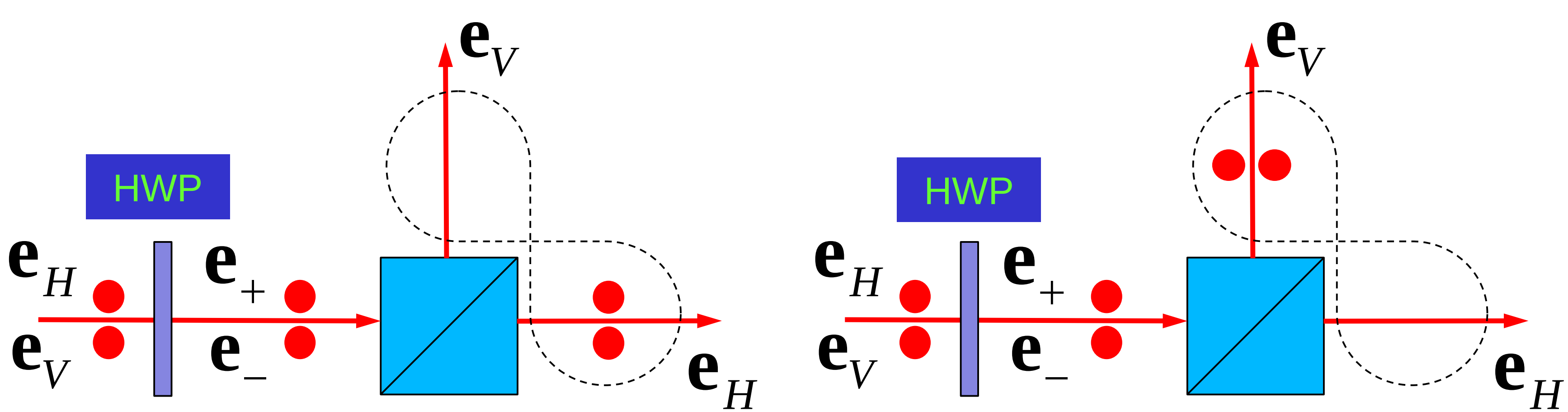
$$\Rightarrow \quad b_+\hat{\mathbf{e}}_+ + b_-\hat{\mathbf{e}}_- = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2$$

Polarization Hong-Ou-Mandel Effect

Input-output of a photon pair on a beam splitter

$$|1\rangle_{\mathbf{e}_+}|1\rangle_{\mathbf{e}_-} = a^\dagger_{\mathbf{e}_+} a^\dagger_{\mathbf{e}_-}|\text{vac}\rangle = \left(\frac{b^\dagger_{\mathbf{e}_H} + b^\dagger_{\mathbf{e}_V}}{\sqrt{2}}\right)\left(\frac{b^\dagger_{\mathbf{e}_H} - b^\dagger_{\mathbf{e}_V}}{\sqrt{2}}\right)|\text{vac}\rangle = \left(\frac{\left(b^\dagger_{\mathbf{e}_H}\right)^2 - \left(b^\dagger_{\mathbf{e}_V}\right)^2}{2}\right)|\text{vac}\rangle$$

$$|1\rangle_{\mathbf{e}_+}|1\rangle_{\mathbf{e}_-} = \frac{|2\rangle_{\mathbf{e}_H}|0\rangle_{\mathbf{e}_V} - |0\rangle_{\mathbf{e}_H}|2\rangle_{\mathbf{e}_V}}{\sqrt{2}}$$



Two-mode squeezed states

$$S_{\mu\nu}(\xi) = \exp(\xi^* a_\mu b_\nu - \xi a_\mu^\dagger b_\nu^\dagger) \quad \left(\xi = |\xi| e^{i\theta} \in \mathbb{C}\right)$$

$$S_{\mu\nu}(\xi)\left|0\right\rangle_\mu \left|0\right\rangle_\nu = \left|\xi\right\rangle_{\mu\nu}$$

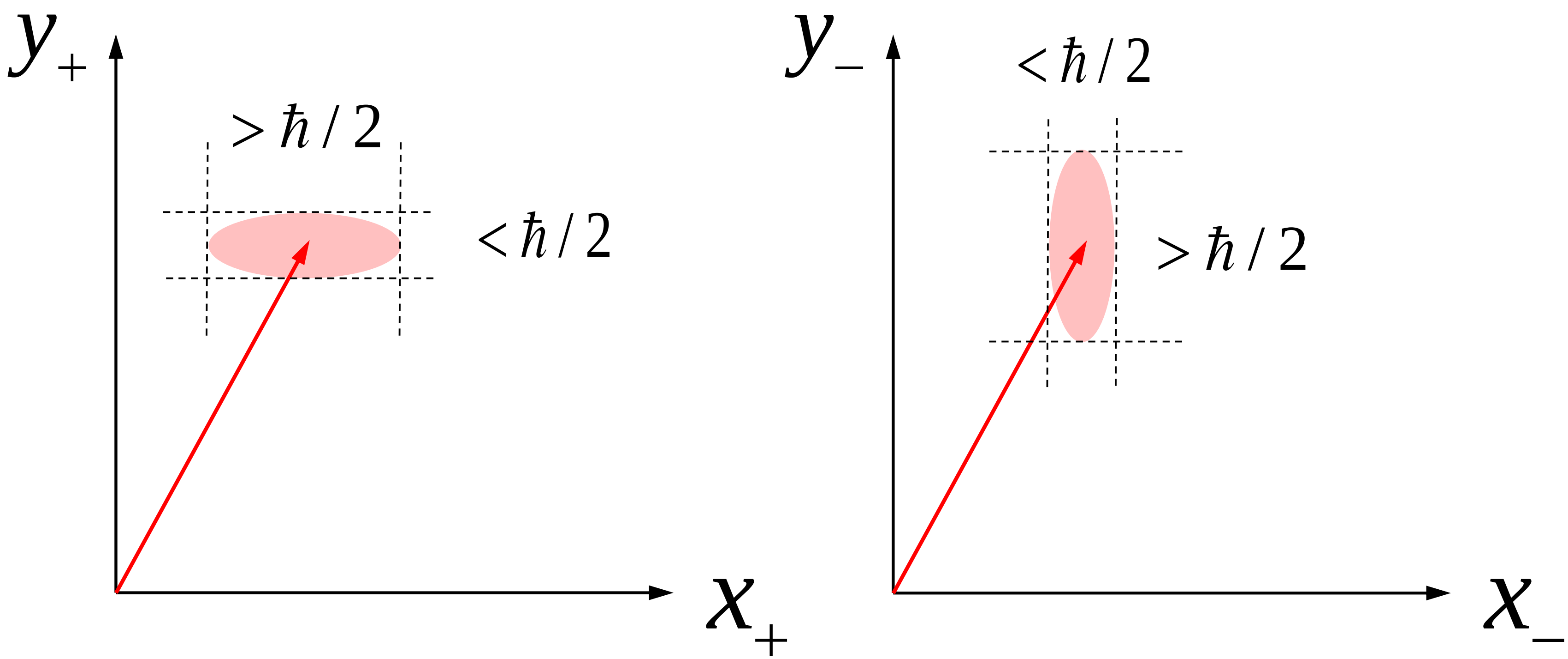
Quadrature entanglement

EPR variables:

$$X_\pm = \frac{X_\mu \pm X_\nu}{2} \quad Y_\pm = \frac{Y_\mu \pm Y_\nu}{2}$$

$$\left\langle (\Delta X_-)^2 \right\rangle + \left\langle (\Delta Y_+)^2 \right\rangle < \hbar \quad \text{(quadrature entanglement)}$$

Two-mode squeezed states

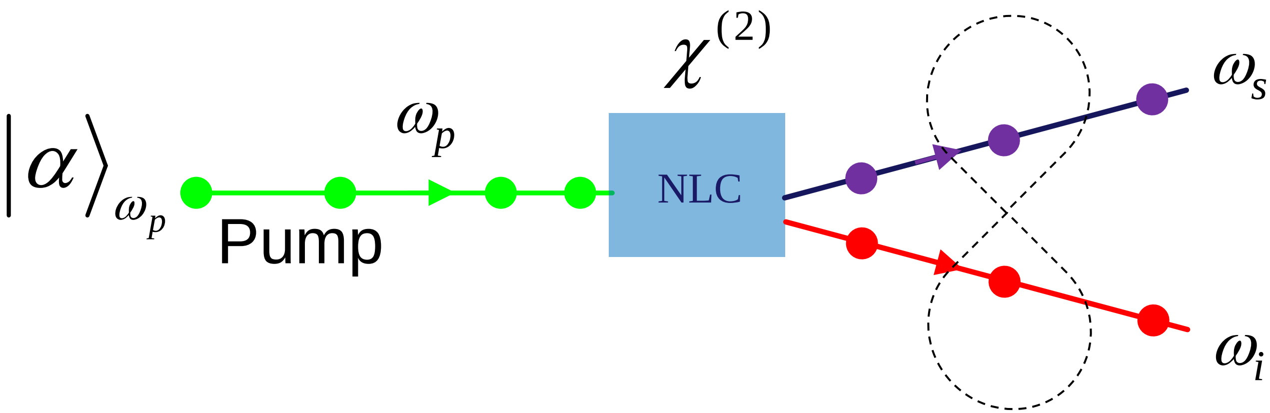


EPR correlations (quadrature entanglement):

$$\left\langle \left(\Delta X_-\right)^2 \right\rangle + \left\langle \left(\Delta Y_+\right)^2 \right\rangle < \hbar$$

Two-mode squeezed state generation

- Nondegenerate parametric down-conversion



$$H_I = i\chi^{(2)}\left(a_{\omega_p} b^\dagger_{\omega_s} b^\dagger_{\omega_i} - a^\dagger_{\omega_p} b_{\omega_s} b_{\omega_i}\right) \approx i\chi^{(2)}\left(\alpha b^\dagger_{\omega_s} b^\dagger_{\omega_i} - \alpha^* b_{\omega_s} b_{\omega_i}\right)$$

$$S_{\omega_s\omega_i}(\xi)|0\rangle_{\omega_p} = |\xi\rangle_{\omega_s\omega_i}$$

$$\xi = \chi^{(2)}\alpha$$

Quantized vector vortices

Vector vortices

Tensor product in CO

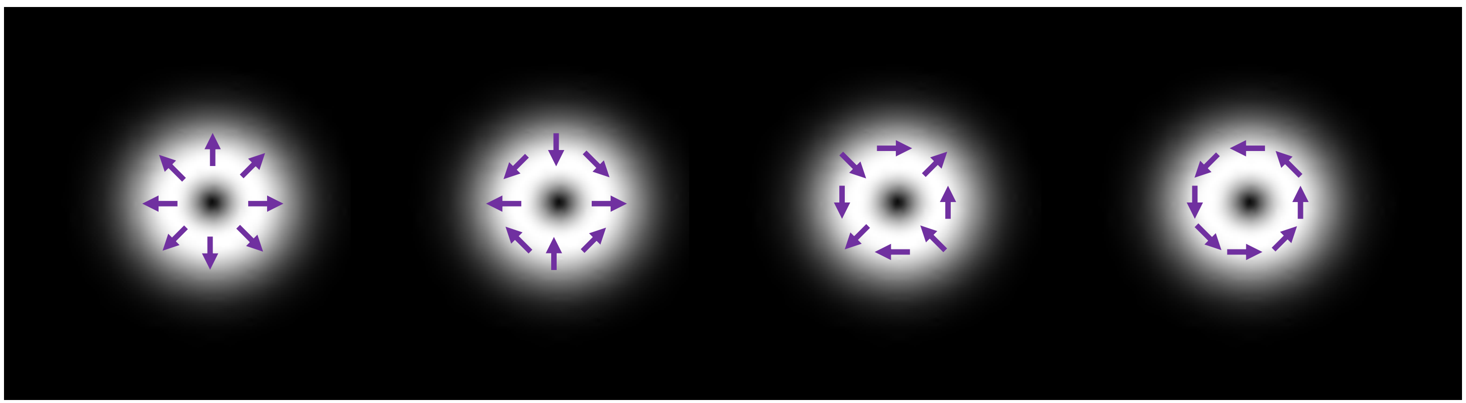
$$\Psi_{sep} = \varphi(\vec{r}) \otimes \hat{\xi} \quad (spatial \otimes polarization)$$

$$\Psi_{ent} \neq \varphi(\vec{r}) \otimes \hat{\xi} \quad \Rightarrow$$

$$\Psi^{\pm} = \frac{1}{\sqrt{2}}\left[\psi_H(\vec{r})\hat{e}_H \pm \psi_V(\vec{r})\hat{e}_V\right]$$

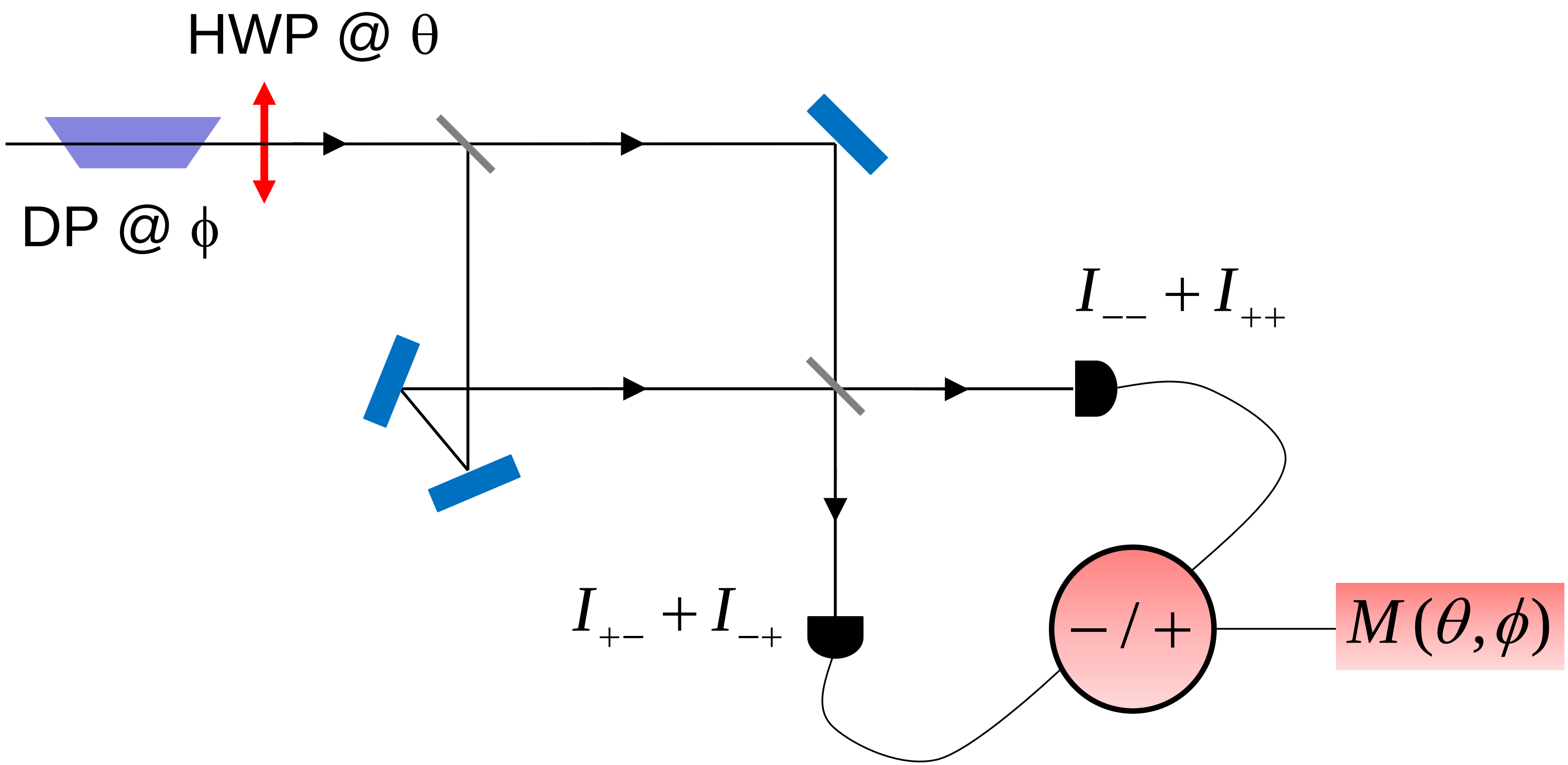
$$\Phi^{\pm} = \frac{1}{\sqrt{2}}\left[\psi_H(\vec{r})\hat{e}_V \pm \psi_V(\vec{r})\hat{e}_H\right]$$

Bell modes



Ψ^{+} Ψ^{-} Φ^{+} Φ^{-}

Bell measurement



Projected intensities

$$I_{\pm\pm}(\theta,\phi) = \left| \int \left[\hat{\mathbf{e}}^*_{\theta\pm} \cdot \mathbf{\Psi}(\mathbf{r}) \right] \psi^*_{\phi\pm}(\mathbf{r})\, d^2\mathbf{r} \right|^2$$

Bell-like Inequality for Spin-Orbit Separability

Projected intensities

$$I_{\pm\pm}(\theta,\phi) = \left| \int \left[\hat{\mathbf{e}}^*_{\theta\pm} \cdot \boldsymbol{\Psi}(\mathbf{r}) \right] \psi^*_{\phi\pm}(\mathbf{r})\, d^2\mathbf{r} \right|^2$$

$$M(\theta,\phi) = I_{++}(\theta,\phi) + I_{--}(\theta,\phi) - I_{+-}(\theta,\phi) - I_{-+}(\theta,\phi)$$

$$S = M\left(\frac{\pi}{16},0\right) + M\left(\frac{\pi}{16},\frac{\pi}{8}\right) - M\left(\frac{3\pi}{16},0\right) + M\left(\frac{3\pi}{16},\frac{\pi}{8}\right)$$

$S < 2 \leftarrow$ separable

$S > 2 \rightarrow$ nonseparable

$S = 2\sqrt{2} \rightarrow$ Bell – like

Quantized vector vortices

$$\mathbf{E}(\mathbf{r},t)=a_{\Psi^+}\mathbf{\Psi}^+(\mathbf{r})+a_{\Psi^-}\mathbf{\Psi}^-(\mathbf{r})+a_{\Phi^+}\mathbf{\Phi}^+(\mathbf{r})+a_{\Phi^-}\mathbf{\Phi}^-(\mathbf{r})$$

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)=&\,a_{HH}(t)\psi_H(\mathbf{r})\hat{\mathbf{e}}_H+a_{HV}(t)\psi_H(\mathbf{r})\hat{\mathbf{e}}_V\\&+a_{VH}(t)\psi_V(\mathbf{r})\hat{\mathbf{e}}_H+a_{VV}(t)\psi_V(\mathbf{r})\hat{\mathbf{e}}_V\end{aligned}$$

$$\mathbf{\Psi}^{\pm}=\frac{1}{\sqrt{2}}\left[\psi_H(\mathbf{r})\hat{\mathbf{e}}_H\pm\psi_V(\mathbf{r})\hat{\mathbf{e}}_V\right]$$

$$a^{\dagger}_{\Psi_\pm}=\frac{a^{\dagger}_{HH}\pm a^{\dagger}_{VV}}{\sqrt{2}}$$

$$\mathbf{\Phi}^{\pm}=\frac{1}{\sqrt{2}}\left[\psi_H(\mathbf{r})\hat{\mathbf{e}}_V\pm\psi_V(\mathbf{r})\hat{\mathbf{e}}_H\right]$$

$$a^{\dagger}_{\Phi_\pm}=\frac{a^{\dagger}_{HV}\pm a^{\dagger}_{VH}}{\sqrt{2}}$$

Single photon vector vortices

$HH\,/\,VV$ single photon states

$$|1\rangle_{HH}|0\rangle_{VV} = a^{\dagger}_{HH}|\mathrm{vac}\rangle$$
$$|0\rangle_{HH}|1\rangle_{VV} = a^{\dagger}_{VV}|\mathrm{vac}\rangle$$

Separable

$\mathbf{\Psi}^{\pm}$ Fock states

$$|1\rangle_{\Psi^+}|0\rangle_{\Psi^-} = a^{\dagger}_{\Psi^+}|\mathrm{vac}\rangle = \frac{|1\rangle_{HH}|0\rangle_{VV} + |0\rangle_{HH}|1\rangle_{VV}}{\sqrt{2}}$$

$$|1\rangle_{\Psi^+}|1\rangle_{\Psi^-} = a^{\dagger}_{\Psi^+}a^{\dagger}_{\Psi^-}|\mathrm{vac}\rangle = \frac{|2\rangle_{HH}|0\rangle_{VV} - |0\rangle_{HH}|2\rangle_{VV}}{\sqrt{2}}$$

Separable

Bell state

Coherent state vector vortices

HH / *VV* coherent states

$$D_{HH}(\alpha) = \exp\left(\alpha\, a_{HH}^\dagger - \alpha^* a_{HH}\right) \qquad |\alpha\rangle_{HH}|0\rangle_{VV} = D_{HH}(\alpha)|\text{vac}\rangle$$

$$D_{VV}(\alpha) = \exp\left(\alpha\, a_{VV}^\dagger - \alpha^* a_{VV}\right) \qquad |0\rangle_{HH}|\alpha\rangle_{VV} = D_{VV}(\alpha)|\text{vac}\rangle$$

Separable

$\Psi^\pm$ coherent states

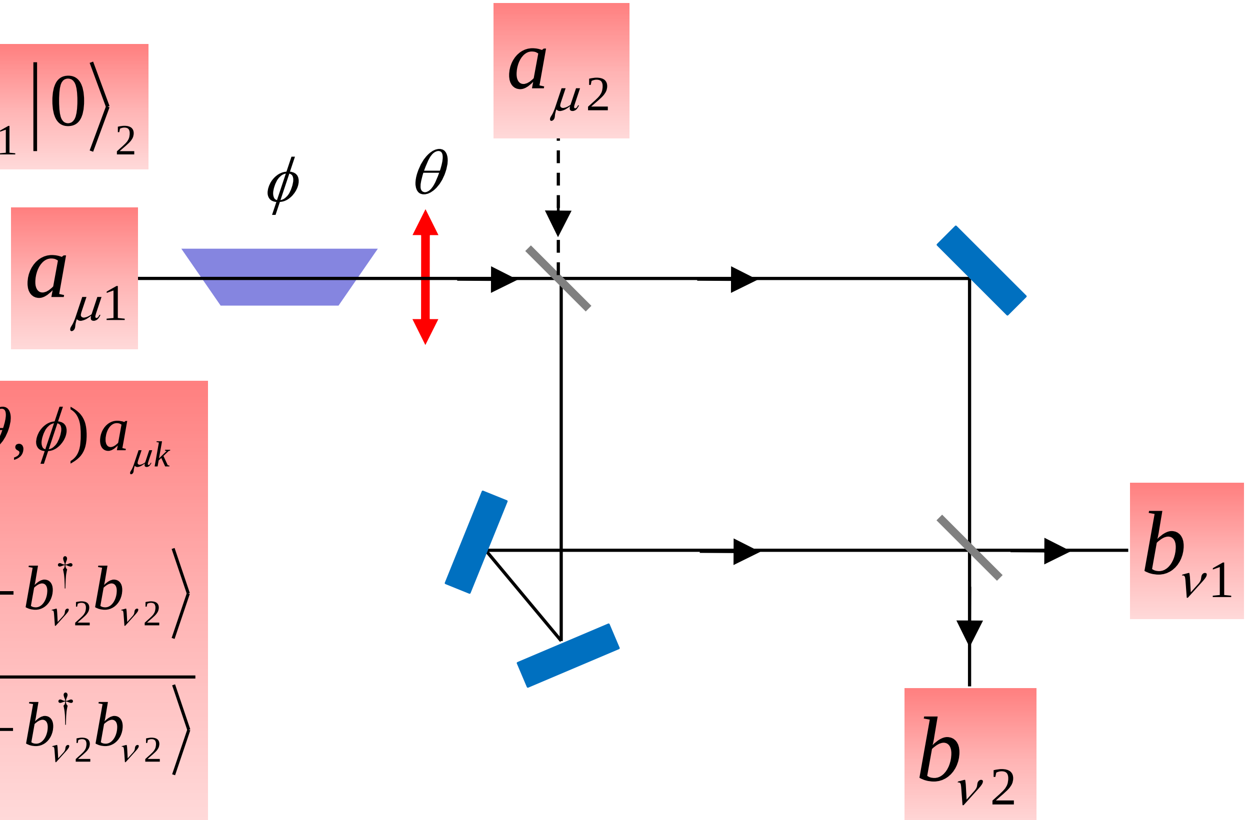
$$D_{\Psi^+}(\alpha) = \exp\left(\alpha\, a_{\Psi^+}^\dagger - \alpha^* a_{\Psi^+}\right) = D_{HH}\left(\alpha/\sqrt{2}\right) D_{VV}\left(\alpha/\sqrt{2}\right)$$

$$|\alpha\rangle_{\Psi^+}|0\rangle_{\Psi^-} = D_{\Psi^+}(\alpha)|\text{vac}\rangle = \left|\alpha/\sqrt{2}\right\rangle_{HH}\left|\alpha/\sqrt{2}\right\rangle_{VV}$$

Separable!!

Bell measurement

Initial state: $|\psi_{in}\rangle_1 |0\rangle_2$



$$b_{\nu j}(\theta,\phi) = \sum_{\mu,k} U_{\nu\mu,jk}(\theta,\phi)\, a_{\mu k}$$

$$M(\theta,\phi) = \frac{\sum_{\nu} \langle b_{\nu 1}^{\dagger} b_{\nu 1} - b_{\nu 2}^{\dagger} b_{\nu 2} \rangle}{\sum_{\nu} \langle b_{\nu 1}^{\dagger} b_{\nu 1} + b_{\nu 2}^{\dagger} b_{\nu 2} \rangle}$$

$$|\psi_{in}\rangle = |1\rangle_{\Psi_+} |0\rangle_{\Psi_-} = \frac{|1\rangle_{HH} |0\rangle_{VV} + |0\rangle_{HH} |1\rangle_{VV}}{\sqrt{2}} \Rightarrow S = 2\sqrt{2}$$

Entangled

$$|\psi_{in}\rangle = |\alpha\rangle_{\Psi_+} |0\rangle_{\Psi_-} = |\alpha/\sqrt{2}\rangle_{HH} |\alpha/\sqrt{2}\rangle_{VV} \Rightarrow S = 2\sqrt{2}$$

Product

Partial entanglement

$$\left|N\right\rangle_{\Psi_+}\left|0\right\rangle_{\Psi_-}=\sum_{n=0}^{N}\sqrt{\frac{N!}{2^N n!(N-n)!}}\left|n\right\rangle_{HH}\left|N-n\right\rangle_{VV}\Rightarrow S=2\sqrt{2}$$

$$\rho_N=\sum_{n=0}^{N}\frac{N!}{2^N n!(N-n)!}\left|n\right\rangle\left\langle n\right|_{HH}\otimes\left|N-n\right\rangle\left\langle N-n\right|_{VV}\Rightarrow S=2\sqrt{2}$$

$$\rho_N(p)=p\left|N\right\rangle\left\langle N\right|_{\Psi_+}\otimes\left|0\right\rangle\left\langle 0\right|_{\Psi_-}+\left(1-p\right)\rho_N\Rightarrow S=\left(1+p\right)\sqrt{2}$$

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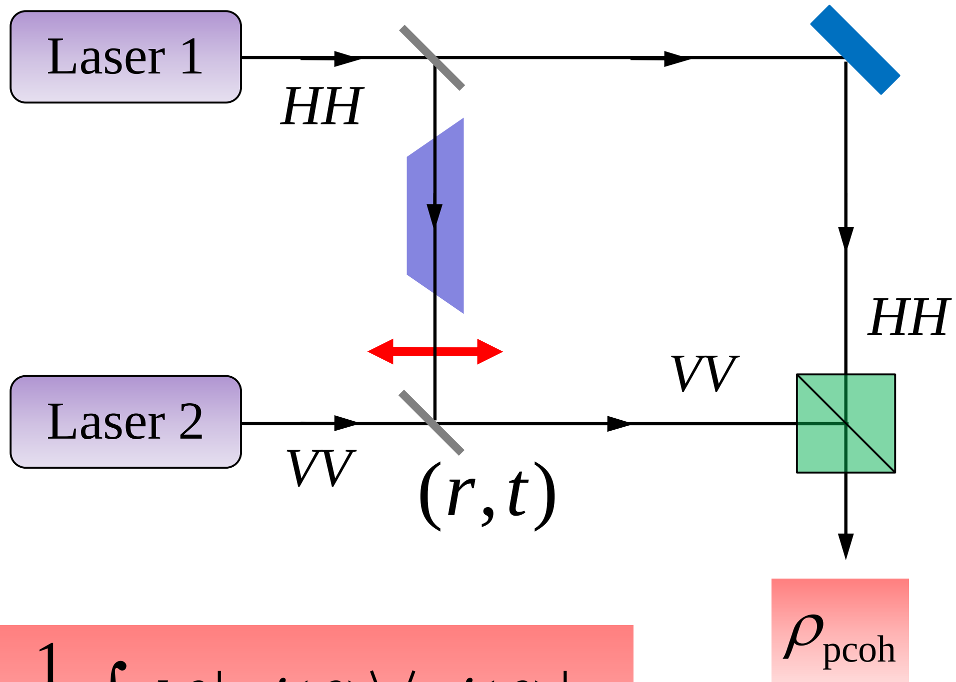
Partial coherence



$$\rho_{\text{pcoh}} = |\alpha\rangle\langle\alpha|_{HH} \otimes \frac{1}{2\pi}\int d\theta\, |\alpha'(\theta)\rangle\langle\alpha'(\theta)|_{VV}$$

$$\alpha'(\theta) = \alpha\left(r + t\, e^{i\theta}\right) \Rightarrow S = (1+r)\sqrt{2}$$

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Conclusions

- Entanglement is independent of the state space basis
- However, it DOES depend on the mode decomposition (partition)
- Fock states can be entangled in one mode structure but not in another
- Coherent states are ALWAYS factorized in any mode structure (classicality)
- Coherent vector beams are “classically entangled” but quantum factorized