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Entangled structures in classical and quantum optics

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Outline

Lecture 1:

Optical vortices as entangled structures in classical and quantum optics

Lecture 2:

Quantum-like simulations and the role of quantum inequalities

Lecture 3:

Quantization of the electromagnetic field

Lecture 4:

Vector beam quantization and the unified framework

Outline

Lecture 3:

Quantization of the electromagnetic field

Electromagnetic modes as harmonic oscillators

Wave Equation in vacuum

Vector potential
Lorentz gauge

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

$$\mathbf{A}(\mathbf{r},t) = A(t)\,\mathbf{u}(\mathbf{r})$$

$$\nabla^2 \mathbf{u} + k^2 \mathbf{u} = 0$$

$$\frac{\partial^2 A}{\partial t^2} + \omega^2 A = 0 \quad (\omega = ck)$$

No charges or currents

$$\rho(\mathbf{r}) = 0$$

$$\mathbf{J}(\mathbf{r}) = \mathbf{0}$$

Scalar potential:

$$\varphi(\mathbf{r}) = 0$$

Helmholtz

Harmonic oscillator

Helmholtz Equation: spatial mode structure

Helmholtz	Spatial modes vector space
$\nabla^2 u_j + k^2 u_j = 0$	$u_j(\mathbf{r}) \in \{\varphi_{lmn}(\mathbf{r})\}$
$\left\{e^{i\mathbf{k}\cdot\mathbf{r}}\right\} \quad (k_x, k_y, k_z)$	Plane
$\left\{B_l(k_\rho \rho) e^{i(k_z z + l\phi)}\right\} \quad (l, k_\rho, k_z)$	Cylindrical
$\left\{b_l(kr) Y_l^m(\theta, \phi)\right\} \quad (l, m, k)$	Spherical
$\left\{LG_{pl}\, e^{ik_z z}\right\} \quad (p, l, k_z)$	Paraxial: LG
$\left\{HG_{mn}\, e^{ik_z z}\right\} \quad (m, n, k_z)$	Paraxial: HG

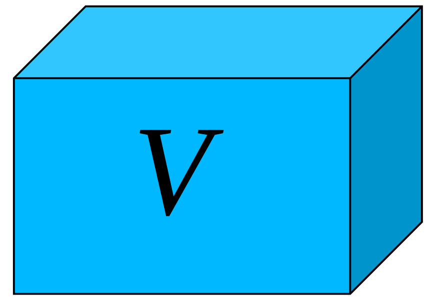
Further requirements

Physical fields

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Boundary conditions



Vector modes: Polarization-Spatial

$$\left\{\mathbf{u}_{jlmn}(\mathbf{r})\right\} \equiv \left\{\mathbf{u}_{\mu}(\mathbf{r})\right\}$$

$$\int_V \mathbf{u}_{\mu}^{*}(\mathbf{r}) \cdot \mathbf{u}_{\nu}(\mathbf{r})\, d^3\mathbf{r} = \delta_{\mu\nu}$$

Compact index

$$(j, l, m, n) \to \mu$$

Finite $V \longrightarrow$ discrete modes

Electromagnetic fields

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mu} A_{\mu}(t)\,\mathbf{u}_{\mu}(\mathbf{r})$$

$$\mathbf{E}(\mathbf{r},t) = -\sum_{\mu} \frac{dA_{\mu}}{dt}\,\mathbf{u}_{\mu}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r},t) = \sum_{\mu} A_{\mu}(t)\,\nabla \times \mathbf{u}_{\mu}$$

Hamiltonian

$$H = \int_{V} \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) d^3\mathbf{r}$$

Time evolution

Quadratures

$$\frac{\partial^2 A_\mu}{\partial t^2} + \omega_\mu^2 A_\mu = 0 \Rightarrow A_\mu(t) = \sqrt{\frac{\omega_\mu}{V\varepsilon_0}}\left(X_\mu \cos\omega_\mu t + Y_\mu \sin\omega_\mu t\right)$$

$$H = \int_V \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right) d^3\mathbf{r} = \sum_\mu H_\mu$$

$$H_\mu = \frac{\omega_\mu}{2}\left(X_\mu^2 + Y_\mu^2\right)$$

Harmonic Oscillator

Canonical quantization

Canonical quantization

Quadratures:
canonical conjugate

$$\left.\begin{array}{l} X_\mu \to \hat{X}_\mu \\ Y_\mu \to \hat{Y}_\mu \end{array}\right\} \Rightarrow \left[\hat{X}_\mu, \hat{Y}_\mu\right] = i\hbar$$

$$a_\mu = \frac{\hat{X}_\mu + i\hat{Y}_\mu}{\sqrt{2\hbar}}$$

Anihilation

$$a_\mu^\dagger = \frac{\hat{X}_\mu - i\hat{Y}_\mu}{\sqrt{2\hbar}}$$

Creation

$$N_\mu = a_\mu^\dagger a_\mu$$

Number

$$\left[a_\mu, a_\nu^\dagger\right] = \mathbf{1}$$

Fock states

$$N_\mu \left|n\right\rangle_\mu = n\left|n\right\rangle_\mu \quad \left(n \in \mathbb{N}\right)$$

$$a_\mu \left|n\right\rangle_\mu = \sqrt{n}\left|n-1\right\rangle_\mu$$

$$a_\mu^\dagger \left|n\right\rangle_\mu = \sqrt{n+1}\left|n+1\right\rangle_\mu$$

Quantum Hamiltonian

$$H_\mu = \hbar\omega_\mu \left(a_\mu^\dagger a_\mu + \frac{1}{2} \right)$$

Single mode

$$H_\mu |n\rangle_\mu = E_{n,\mu} |n\rangle_\mu$$

Eigenvectors: Fock states

$$E_{n,\mu} = \hbar\omega_\mu \left(n + \frac{1}{2} \right)$$

Eigenvalues

Heisenberg Equations

$$\frac{da_\mu}{dt} = \frac{1}{i\hbar}\left[a_\mu, H_\mu\right] = -i\omega_\mu a_\mu \quad \Rightarrow \quad \begin{cases} a_\mu(t) = a_\mu(0)\, e^{-i\omega_\mu t} \\ a_\mu^\dagger(t) = a_\mu^\dagger(0)\, e^{i\omega_\mu t} \end{cases}$$

$$\hat{X}_\mu(t) = \hat{X}_\mu(0)\cos\omega t + \hat{Y}_\mu(0)\sin\omega t$$
$$\hat{Y}_\mu(t) = -\hat{X}_\mu(0)\sin\omega t + \hat{Y}_\mu(0)\cos\omega t$$

Coherent states:

$$a_\mu|\alpha\rangle_\mu = \alpha|\alpha\rangle_\mu \quad \left(\alpha = |\alpha| e^{i\theta} \in \mathbb{C}\right)$$

"Classical-like"

$$x_\mu(t) = \langle\alpha|\hat{X}_\mu(t)|\alpha\rangle = |\alpha|\cos\left(\omega t - \theta\right)$$
$$y_\mu(t) = \langle\alpha|\hat{Y}_\mu(t)|\alpha\rangle = |\alpha|\sin\left(\omega t - \theta\right)$$

Minimum uncertainty

$$\langle\alpha|\left[\Delta\hat{X}_\mu(t)\right]^2|\alpha\rangle = \langle\alpha|\left[\Delta\hat{Y}_\mu(t)\right]^2|\alpha\rangle = \frac{\hbar}{2}$$

Quantum fluctuations

Photon number (energy)

$$\left\langle \left(\Delta N_\mu\right)^2 \right\rangle = \langle \psi | \left(\Delta N_\mu\right)^2 | \psi \rangle = \langle \psi | N_\mu{}^2 | \psi \rangle - \langle \psi | N_\mu | \psi \rangle^2$$

Fock states: $\left\langle \left(\Delta N_\mu\right)^2 \right\rangle = 0$

Coherent states: $\left\langle \left(\Delta N_\mu\right)^2 \right\rangle = |\alpha|^2$

Poisson distribution:

$$|\alpha\rangle_\mu = e^{-|\alpha|^2/2} \sum_m \frac{\alpha^m}{\sqrt{m!}} |m\rangle_\mu$$

$$P(n) = \left| \langle n | \alpha \rangle_\mu \right|^2 = e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!}$$

$$\langle N_\mu \rangle = |\alpha|^2 \qquad \sqrt{\left\langle \left(\Delta N_\mu\right)^2 \right\rangle} = |\alpha|^2 = \langle N_\mu \rangle$$

Conclusions

- Wave equation + boundary conditions → vector modes + quadratures
- Canonical quantization → quadrature commutation relations
- Hamiltonian: modes as an ensemble of harmonic oscillators
- Photons as discrete energy excitations on each mode
- Coherent states: "classical-like" → Poisson photon distribution