Instituto de Física

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Entangled structures in classical and quantum optics

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Preface

- Quantum information: Encoding and processing information in physical systems governed by Quantum Mechanics.
- Basic quantum information unit: *qubits*. $\alpha|0\rangle + \beta|1\rangle$
- Operations: Unitary (quantum gates) + measurements → algorithms.
- Informational advantages: Superpositions and entanglement.
- Examples: Quantum cryptography (BB84) and teleportation
- Platforms: photons, trapped ions, superconducting circuits, etc…
- Drawback: Decoherence
- Our "daily bread": Optical vortices and photonic DoFs.

Outline

Lecture 1:

Optical vortices as entangled structures in classical optics

Lecture 2:

Quantum-like simulations and the role of quantum inequalities

Lecture 3:

Quantization of the electromagnetic field

Lecture 4:

Vector beam quantization and the unified framework

Outline

Lecture 1:

Optical vortices as entangled structures in classical optics

Paraxial Equation

(Paraxial Equation)

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\Downarrow$$

$$\vec{E} = u(\vec{r})\,\hat{n}\,e^{-i\omega t}$$

$$\Downarrow$$

$$\nabla^2 u + k^2 u = 0$$

$$\Rightarrow$$

$$u(\vec{r}) = \psi(x, y, z)\,e^{ikz}$$

$$\Uparrow$$

$$\frac{\partial^2 \psi}{\partial z^2} << k\frac{\partial \psi}{\partial z}$$

$$\Uparrow$$

$$\nabla_{\perp}^2 \psi + 2ik\frac{\partial \psi}{\partial z} = 0$$

Fundamental Gaussian Mode

$$\psi_0(\mathbf{r}) = \sqrt{\frac{2}{\pi}}\frac{1}{w(z)}\exp\left(-\frac{x^2+y^2}{w^2(z)}\right)\exp\left(ik\frac{x^2+y^2}{2R(z)}\right)e^{-i\arctan(z/z_R)}$$

Beam width

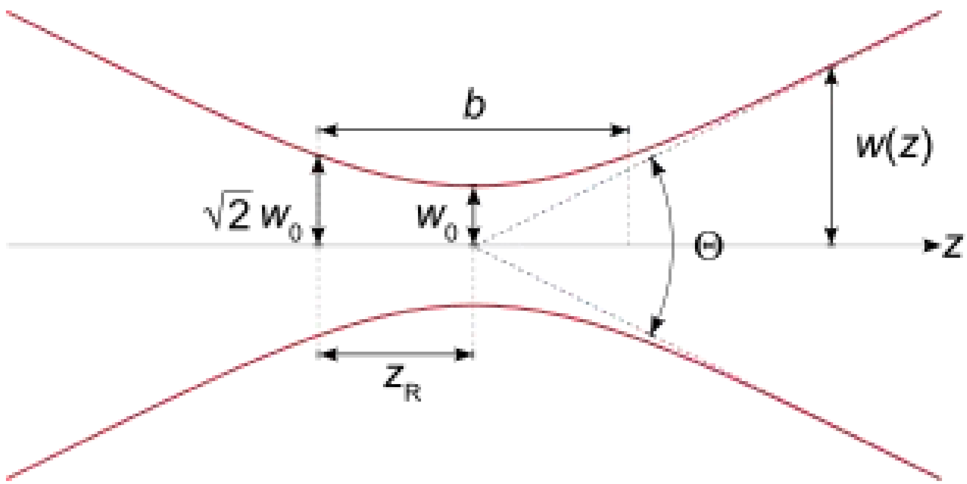
$$w(z) = w_0\sqrt{\left(1+\frac{z^2}{z_R^2}\right)}$$

Wavefront radius

$$R(z) = z\left(1+\frac{z_R^2}{z^2}\right)$$

Rayleigh range

$$z_R = \frac{\pi w_0}{\lambda}$$

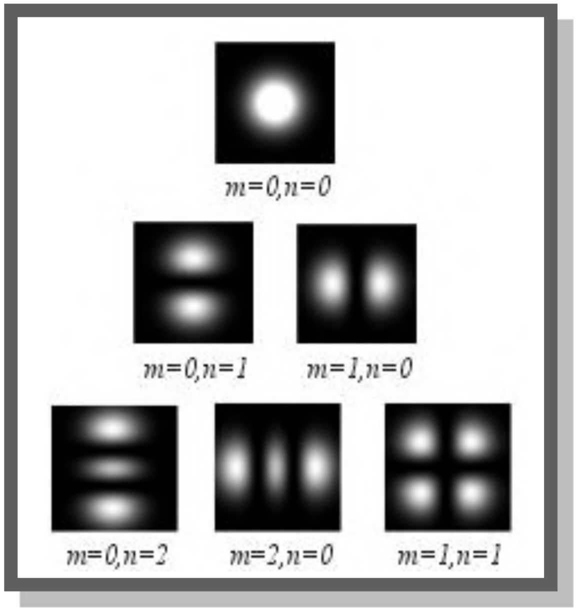


General Paraxial Modes

(Paraxial Equation)

$$\nabla_{\perp}^{2}\psi + 2ik\frac{\partial\psi}{\partial z} = 0$$

Hermite-Gauss (HG)

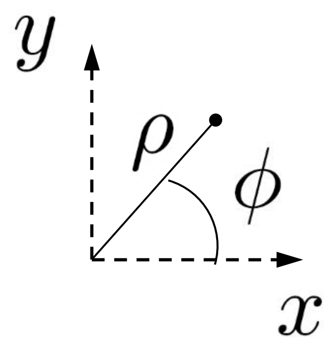


Rectangular

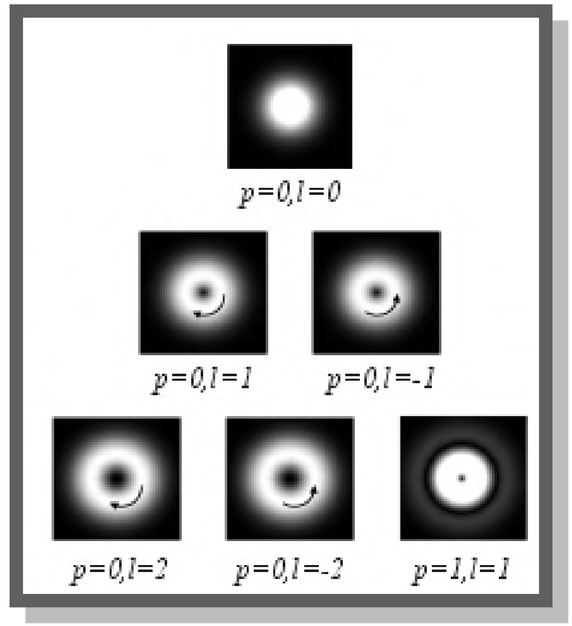
$HG_{nm}(x,y)$

Cylindrical

$LG_{pl}(\rho,\phi)$



Laguerre-Gauss (LG)



Hermite-Gaussian Modes

$$HG_{mn}(\mathbf{r}) = \frac{A_{mn}}{w(z)} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left(-\frac{x^2+y^2}{w^2(z)}\right) \exp\left(ik\frac{x^2+y^2}{2R(z)}\right) e^{-i\varphi_N(z)}$$

Gouy phase

$$\varphi_N(z) = (N+1)\arctan\left(z / z_R\right)$$

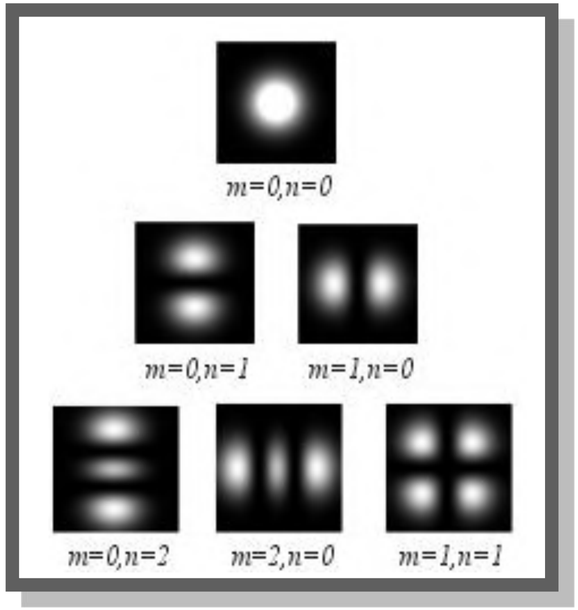
N=m+n

Orthonormal

$$\int HG_{mn}^*(\mathbf{r})\, HG_{m'n'}(\mathbf{r})\, d^2\mathbf{r} = \delta_{mm'}\,\delta_{nn'}$$

Complete

$$\sum_{m,n} HG_{mn}^*(\mathbf{r})\, HG_{mn}(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$



N=0

N=1

N=2

Laguerre-Gaussian Modes

$$LG_{pl}(\mathbf{r}) = \frac{A_{pl}}{w(z)}\left(\frac{\sqrt{2}\rho}{w(z)}\right)^{|l|} L_p^{|l|}\left(\frac{2\rho^2}{w^2(z)}\right)\exp\left(-\frac{\rho^2}{w^2(z)}\right)\exp\left(ik\frac{\rho^2}{2R(z)}\right)e^{-i\varphi_N(z)}e^{il\phi}$$

Gouy phase

$$\varphi_N(z) = (N+1)\arctan(z/z_R)$$

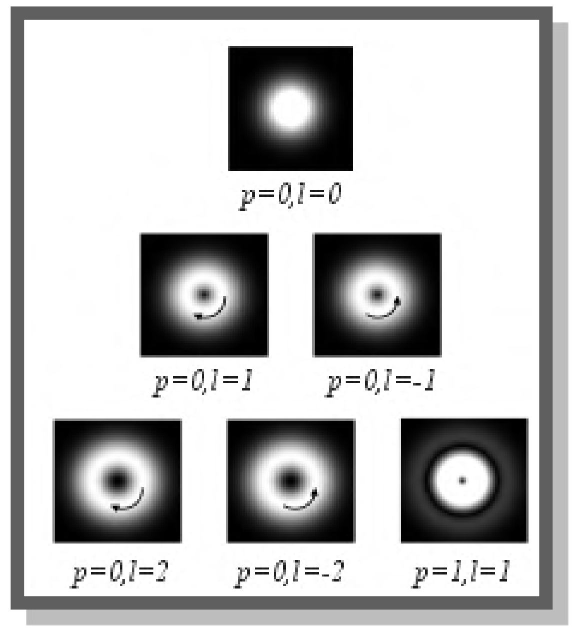
N=2p+|l|

Orthonormal

$$\int LG^*_{pl}(\mathbf{r})\,LG_{p'l'}(\mathbf{r})\,d^2\mathbf{r} = \delta_{pp'}\,\delta_{ll'}$$

Complete

$$\sum_{p,l} LG^*_{pl}(\mathbf{r})\,LG_{pl}(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$



N=0

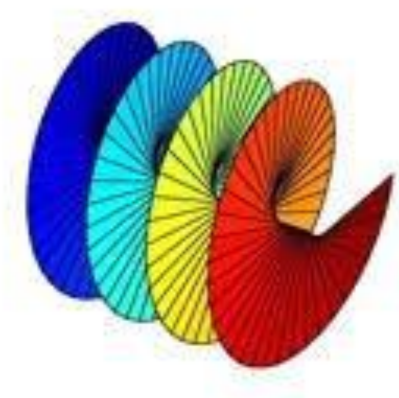
N=1

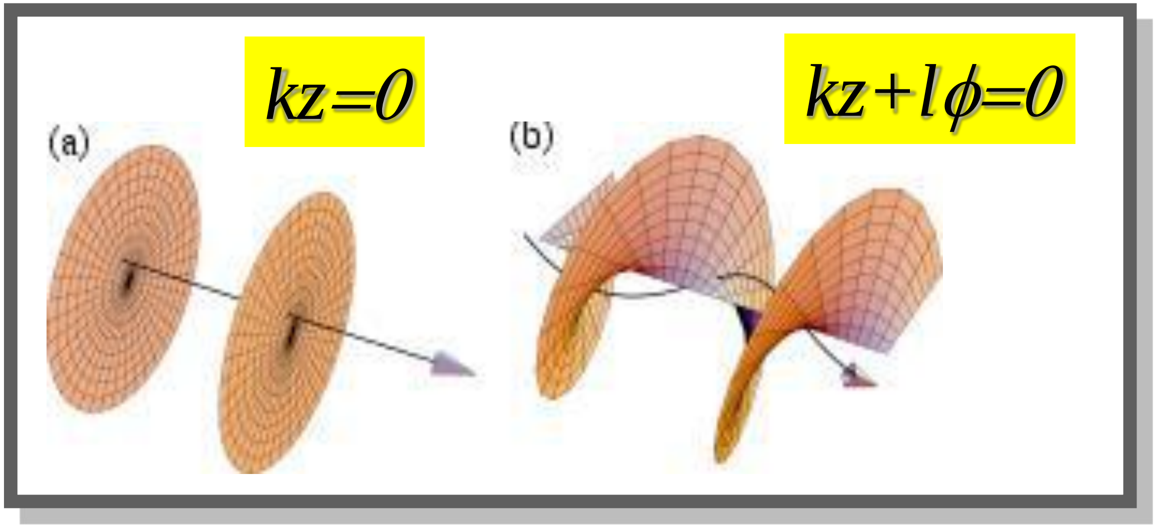
N=2

Orbital angular momentum

Twisted wavefront

l=1





$$\mathbf{P} = \varepsilon_0 \int \mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\, d^3\mathbf{r}$$

Linear momentum

$$\mathbf{J}(\mathbf{r}_0) = \varepsilon_0 \int (\mathbf{r} - \mathbf{r}_0) \times [\mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})]\, d^3\mathbf{r}$$

Angular momentum

Paraxial propagation

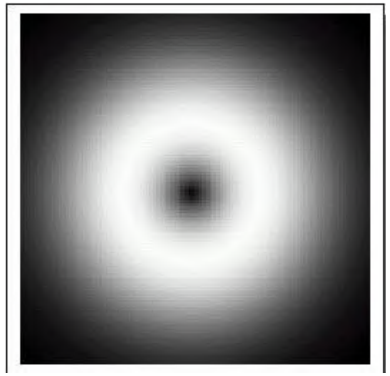
$$\mathbf{J} = \mathbf{J}_S + \mathbf{J}_O$$

SPIN + ORBITAL

pol *wavefront*

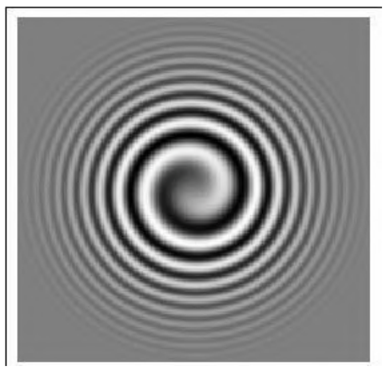
Intensity and phase of LG modes

Intensity



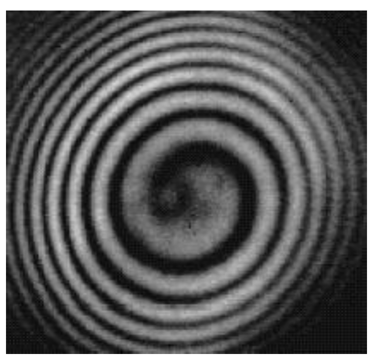
$$\left|LG_{0,\pm 1}(r,\phi)\right|^2 \propto r^2 e^{-2r^2/w^2}$$

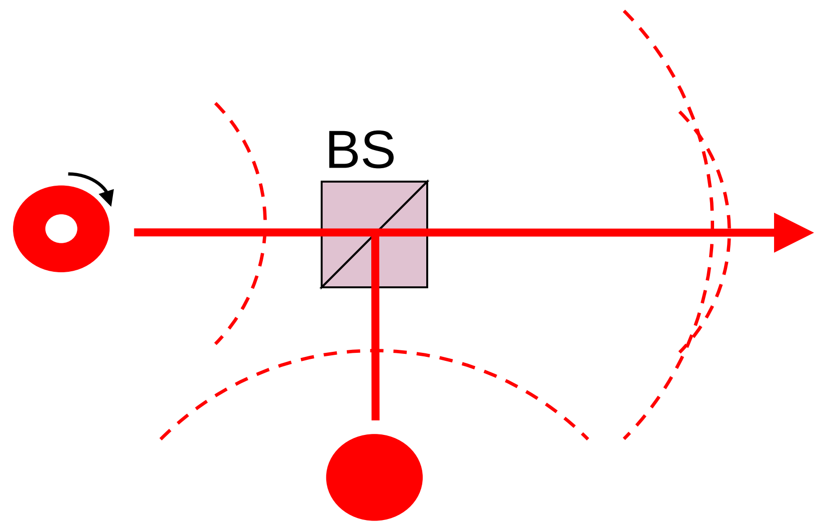
Phase (theo)



$$LG_{0,\pm 1}(r,\phi) \propto r\, e^{\pm i\phi} e^{-r^2/w^2}$$

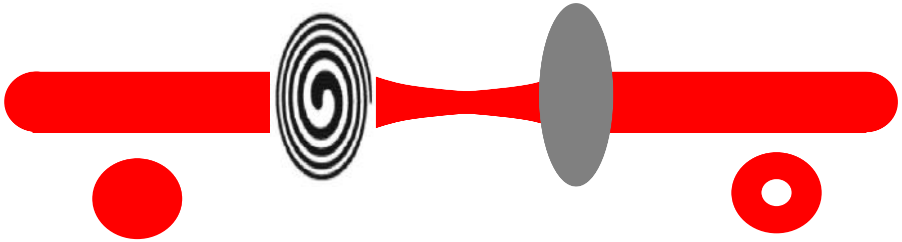
Phase (exp)

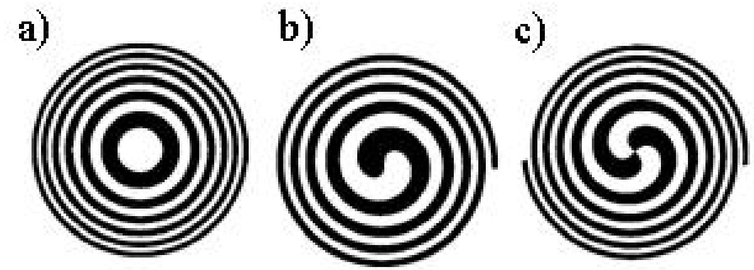




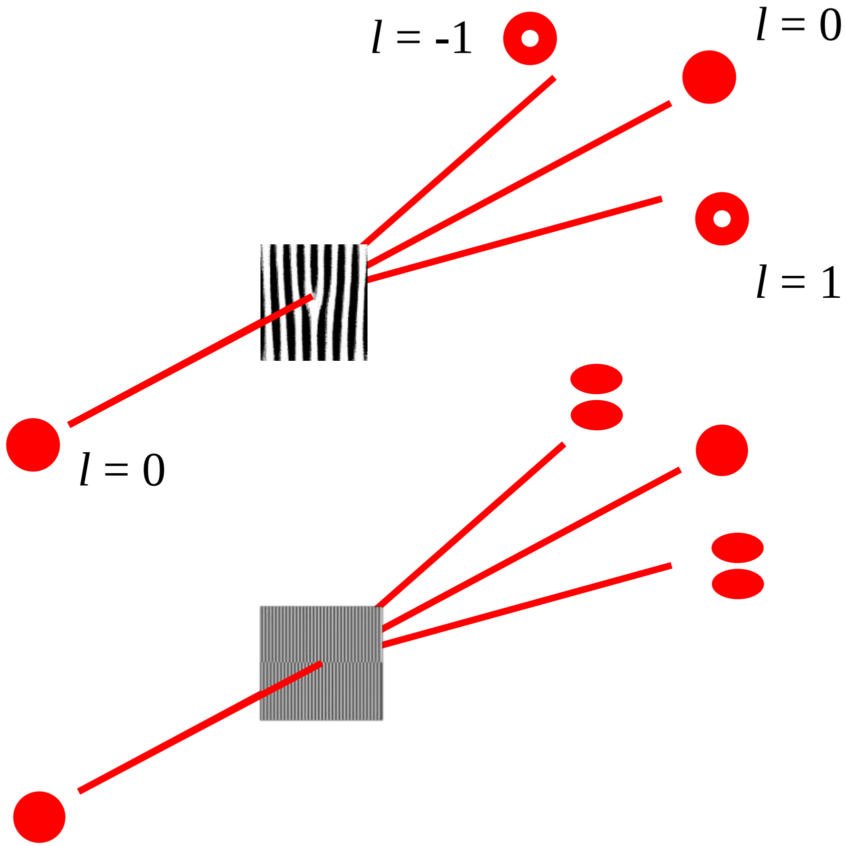
Holographic production of LG and HG beams

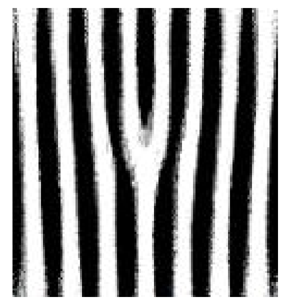
Confocal lens

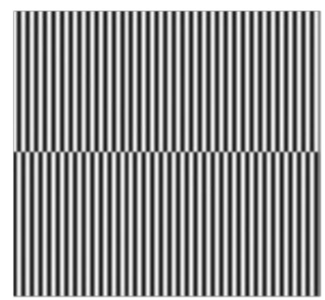


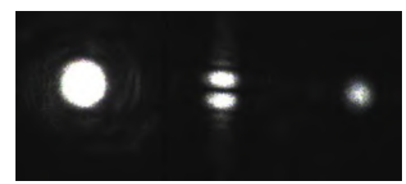


N.R. Heckenberg et al, Opt. Lett. **17**, 221 (1992)









Algebraic structure of paraxial wave functions

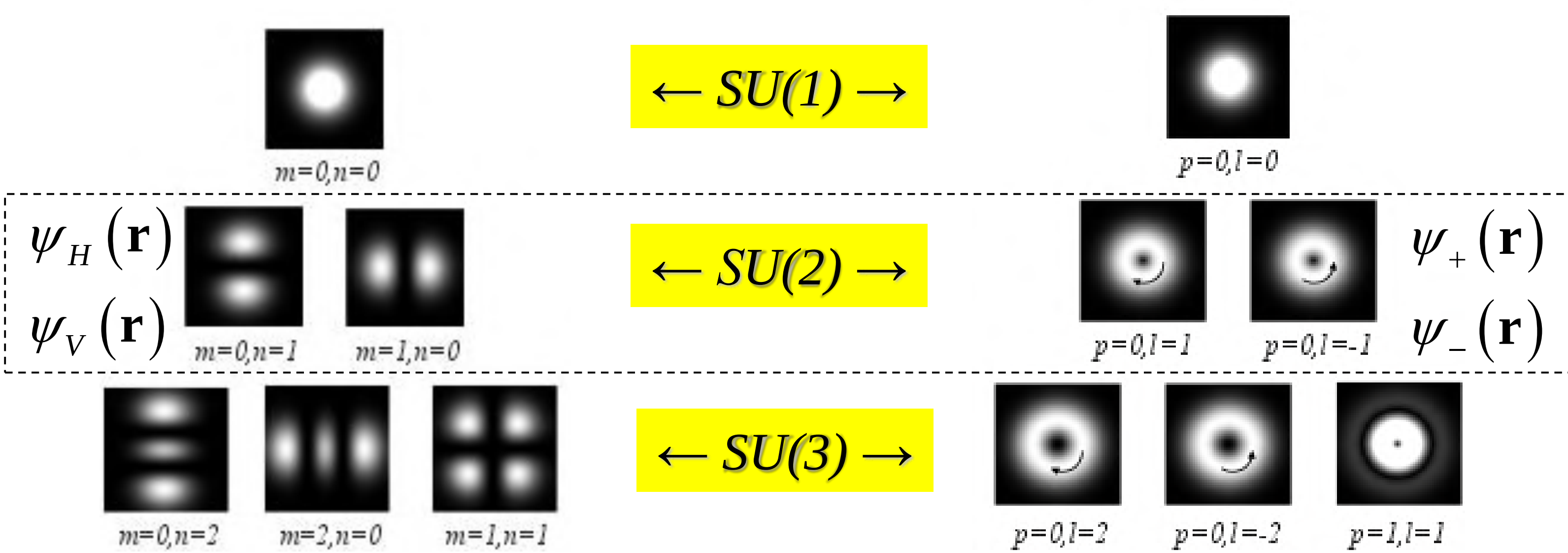
$$LG_{pl}(\mathbf{r}) = \sum_{k=0}^{N} \alpha_k \, HG_{k,N-k}(\mathbf{r})$$

E. Abramochkin and V. Volostnikov,
Opt. Commun. 83, 123 (1991)

LG-HG Unitary transformation

$$U_{nm}^{pl} = \sum_{\oplus} U^{(N)}$$

Hermite-Gauss (HG) **Laguerre-Gauss (LG)**

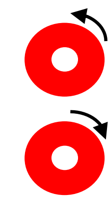


Astigmatic mode transformations

$$\psi_H(\mathbf{r}) \propto x\, e^{-(x^2+y^2)/w^2}$$

$$\psi_V(\mathbf{r}) \propto y\, e^{-(x^2+y^2)/w^2}$$

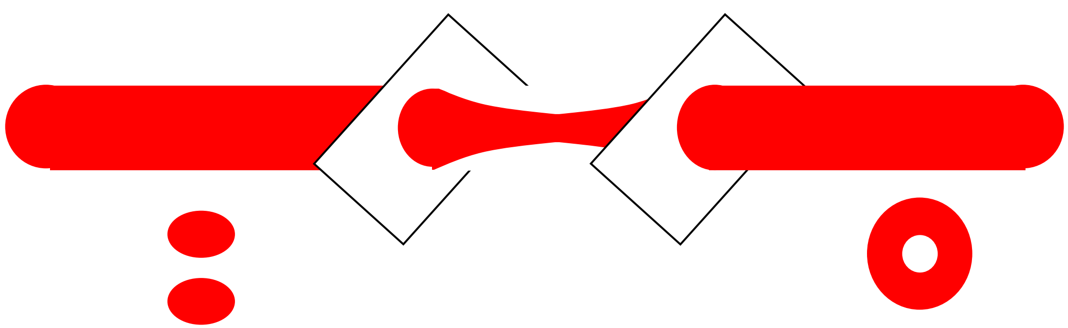
HG-LG $\quad \psi_{\pm} = \dfrac{\psi_H \pm i\,\psi_V}{\sqrt{2}}$

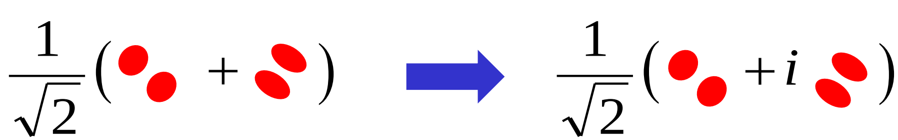


HG-HG $\quad \psi_{\pm 45^\circ} = \dfrac{\psi_H \pm \psi_V}{\sqrt{2}}$

Mode Converter

cylindrical lenses at 45º

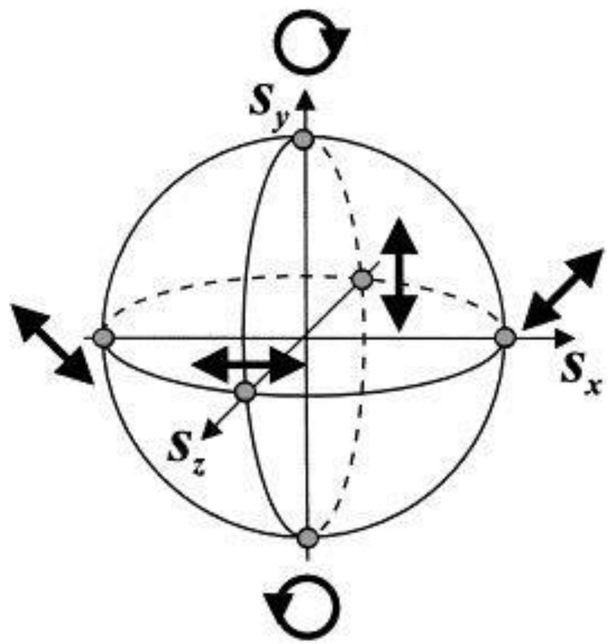


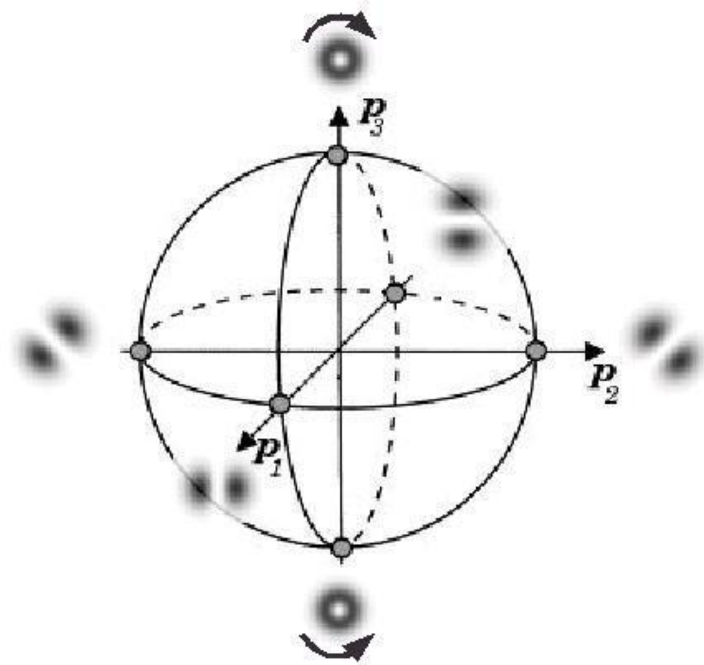


MC eigenvectors

$$\psi_{\pm 45^\circ} = \frac{\psi_H \pm \psi_V}{\sqrt{2}}$$

Poincaré representation





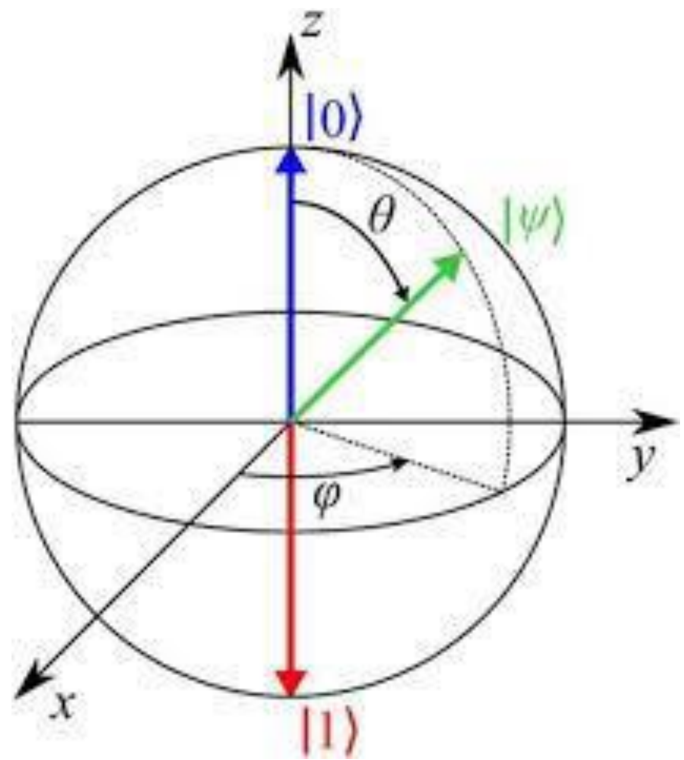
Poincaré sphere for polarization modes

Poincaré sphere for first order modes

$$\hat{\mathbf{e}}_{\theta,\varphi} = \cos\frac{\theta}{2}\,\hat{\mathbf{e}}_H + e^{i\varphi}\sin\frac{\theta}{2}\,\hat{\mathbf{e}}_V \qquad \psi_{\theta,\varphi} = \cos\frac{\theta}{2}\,\psi_H + e^{i\varphi}\sin\frac{\theta}{2}\,\psi_V$$

Quantum computation unit: QUBIT

Bloch sphere



$$\left|\theta,\phi\right\rangle = \cos\frac{\theta}{2}\left|0\right\rangle + e^{i\varphi}\sin\frac{\theta}{2}\left|1\right\rangle$$

Spin-Orbit Entanglement

Spin-Orbit Modes

Tensor product in QM

$$|\psi\rangle_{AB} = |\eta\rangle_A \otimes |\xi\rangle_B$$

$$|\psi\rangle_{AB} \neq |\eta\rangle_A \otimes |\xi\rangle_B$$

Entanglement

$$|\Psi_\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

$$|\Phi_\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

Bell states

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$C = 2|\alpha\delta - \beta\gamma|$$

concurrence

$$0 \leq C \leq 1$$

Tensor product in CO

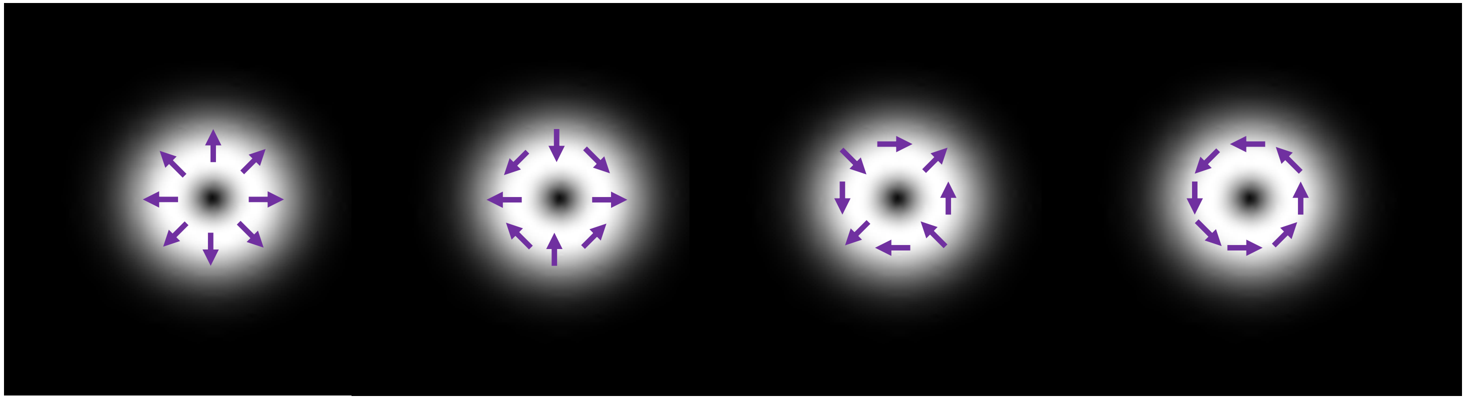
$$\boldsymbol{\Psi}_{sep} = \eta(\mathbf{r}) \otimes \hat{\boldsymbol{\varepsilon}} \qquad (spatial \otimes polarization)$$

$$\boldsymbol{\Psi}_{ent} \neq \eta(\mathbf{r}) \otimes \hat{\boldsymbol{\varepsilon}} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{\Psi}^\pm &= \frac{1}{\sqrt{2}}\left[\psi_H(\mathbf{r})\,\hat{\mathbf{e}}_H \pm \psi_V(\mathbf{r})\,\hat{\mathbf{e}}_V\right] \\ \boldsymbol{\Phi}^\pm &= \frac{1}{\sqrt{2}}\left[\psi_H(\mathbf{r})\,\hat{\mathbf{e}}_V \pm \psi_V(\mathbf{r})\,\hat{\mathbf{e}}_H\right] \end{aligned}$$

Bell modes

$$\boldsymbol{\Psi} = \alpha\,\psi_H(\mathbf{r})\hat{\mathbf{e}}_H + \beta\,\psi_H(\mathbf{r})\hat{\mathbf{e}}_V + \gamma\,\psi_V(\mathbf{r})\hat{\mathbf{e}}_H + \delta\,\psi_V(\mathbf{r})\hat{\mathbf{e}}_V \qquad C = 2|\alpha\delta - \beta\gamma|$$

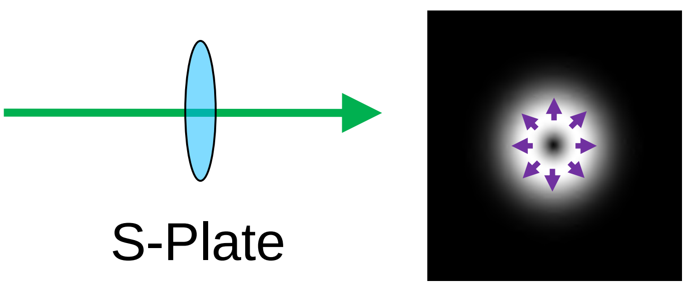
Polarization vortices



Ψ^+ Ψ^- Φ^+ Φ^-

$$\Psi^\pm = \frac{1}{\sqrt{2}}\left[\psi_H(\mathbf{r})\,\hat{\mathbf{e}}_H \pm \psi_V(\mathbf{r})\,\hat{\mathbf{e}}_V\right]$$

$$\Phi^\pm = \frac{1}{\sqrt{2}}\left[\psi_H(\mathbf{r})\,\hat{\mathbf{e}}_V \pm \psi_V(\mathbf{r})\,\hat{\mathbf{e}}_H\right]$$



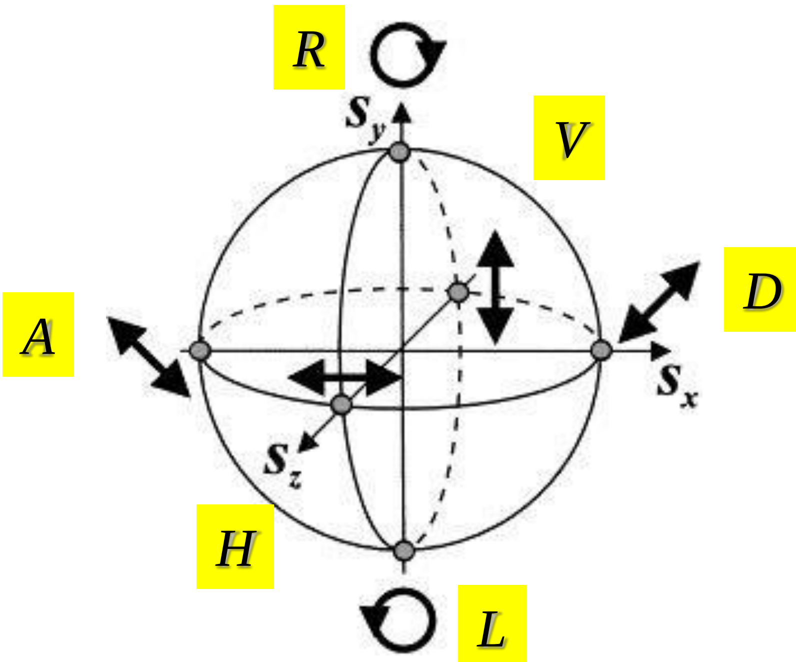
Ψ^+

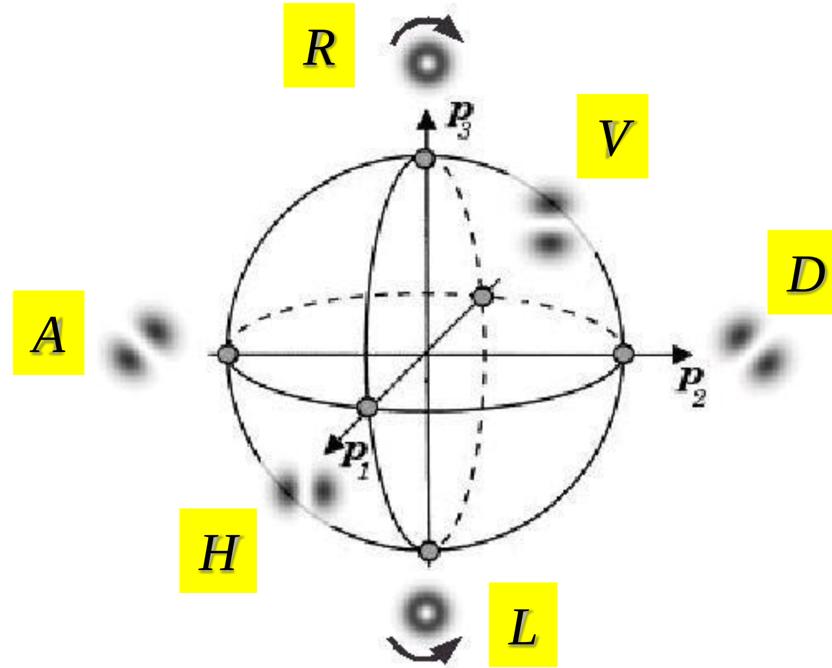
Spin-orbit coupling in liquid crystals

L. Marrucci, C. Manzo, and D. Paparo, Phys. Rev. Lett. 96, 163905 (2006)

State tomography: Polarimetry

Spin and orbital Stokes parameters



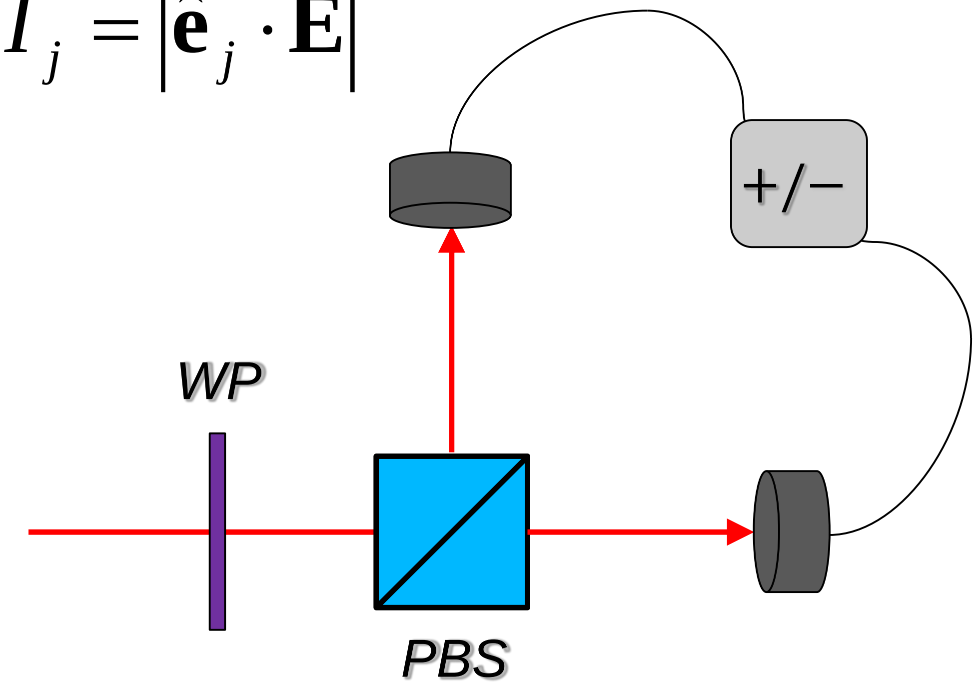


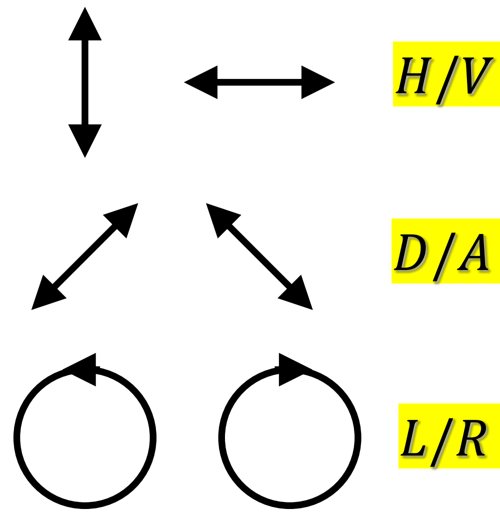
Spin	Orbital
$S_1 = (I_H - I_V)/I_{TOT}$	$O_1 = (I_H - I_V)/I_{TOT}$
$S_2 = (I_D - I_A)/I_{TOT}$	$O_2 = (I_D - I_A)/I_{TOT}$
$S_3 = (I_R - I_L)/I_{TOT}$	$O_3 = (I_R - I_L)/I_{TOT}$

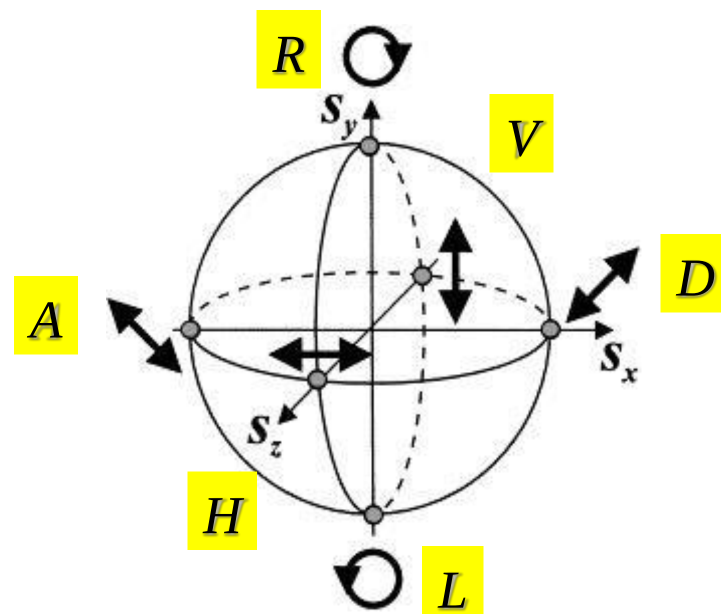
Polarization Stokes Parameters

Polarization projection

$$I_j = \left|\hat{\mathbf{e}}_j^* \cdot \mathbf{E}\right|^2$$



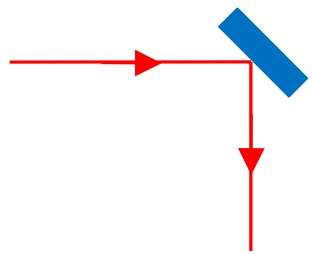




$$S_1 = \frac{I_H - I_V}{I_{TOT}} \quad S_2 = \frac{I_D - I_A}{I_{TOT}} \quad S_3 = \frac{I_L - I_R}{I_{TOT}}$$

$$S_1^2 + S_2^2 + S_3^2 \leq 1$$

Parity mode selector

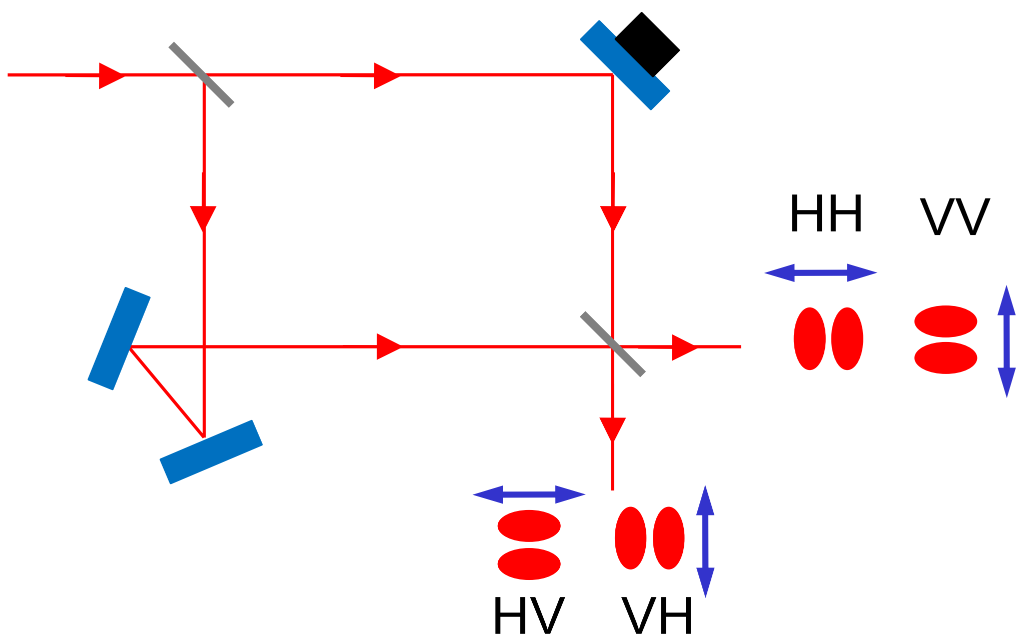


Mirror reflection

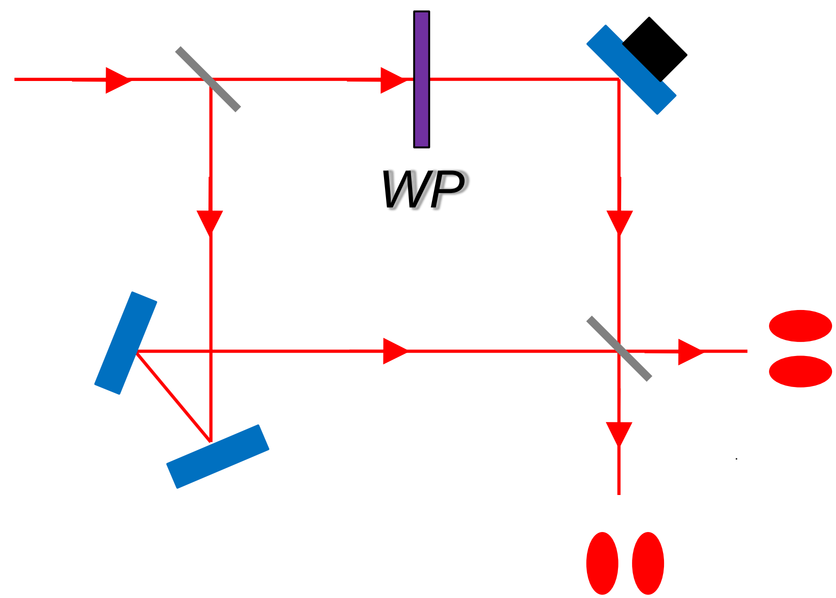
$$\hat{\mathbf{e}}_H \to -\hat{\mathbf{e}}_H \qquad \psi_H \to -\psi_H$$

$$\hat{\mathbf{e}}_V \to +\hat{\mathbf{e}}_V \qquad \psi_V \to +\psi_V$$

Spatial-pol parity



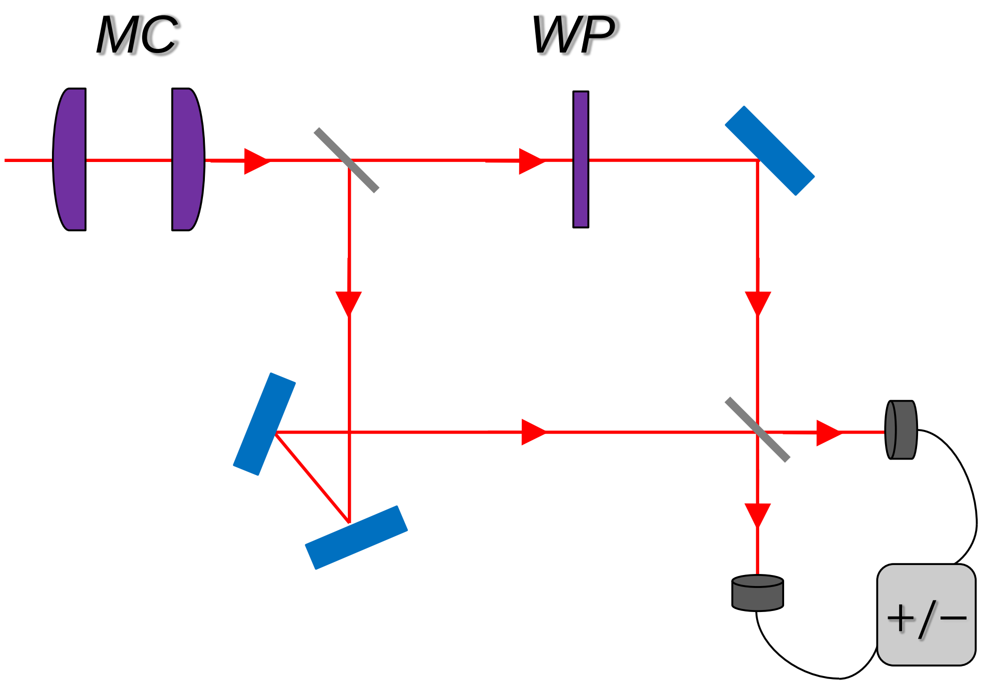
Spatial parity

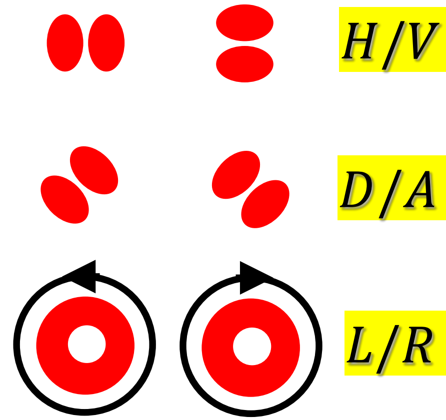


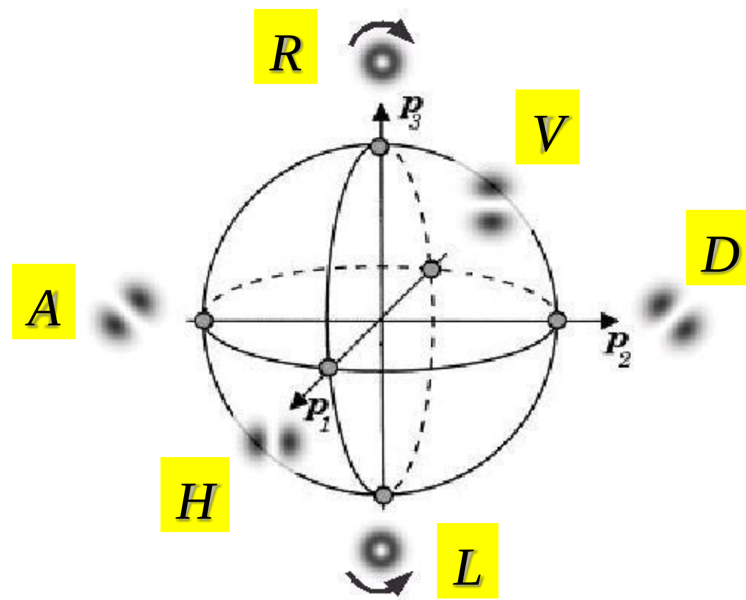
Orbital Stokes Parameters

$$I_j = \left| \int \varphi_j^*(\mathbf{r}) E(\mathbf{r})\, d^2\mathbf{r} \right|^2$$

Spatial mode projection







$$O_1 = \frac{I_H - I_V}{I_{TOT}} \qquad O_2 = \frac{I_D - I_A}{I_{TOT}} \qquad O_3 = \frac{I_L - I_R}{I_{TOT}}$$

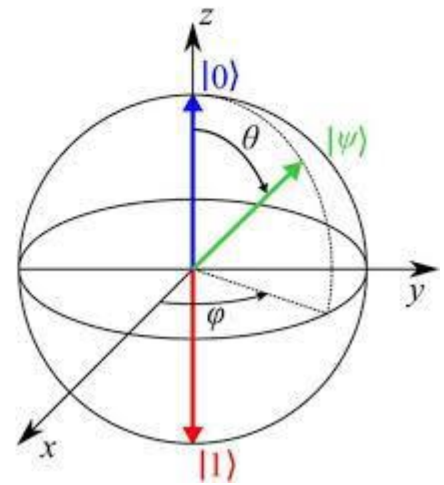
$$O_1^2 + O_2^2 + O_3^2 \leq 1$$

State tomography in QM

General state

$$\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix} = \begin{bmatrix} 1+z/2 & (x-iy)/2 \\ (x+iy)/2 & 1-z/2 \end{bmatrix}$$

$$\rho_{aa}, \rho_{bb} \in \mathbb{R}, \; \rho_{aa} + \rho_{bb} = 1, \; \rho_{ba} = \rho_{ab}^*$$



Mutually unbiased bases

$$\left\{ |+\rangle_z, |-\rangle_z \right\}, \left\{ |+\rangle_x, |-\rangle_x \right\}, \left\{ |+\rangle_y, |-\rangle_y \right\}$$

$$|\pm\rangle_x = \frac{|+\rangle_z \pm |-\rangle_z}{\sqrt{2}} \qquad |\pm\rangle_y = \frac{|+\rangle_z \pm i|-\rangle_z}{\sqrt{2}}$$

$$\sigma_j |\pm\rangle_j = \pm |\pm\rangle_j \qquad \left| {}_j\langle \pm | \pm \rangle_k \right| = \frac{1}{\sqrt{2}} \; (j \neq k)$$

Tomographic measurements

$$x = \rho_{ab} + \rho_{ba} = Tr(\rho\sigma_x)$$

$$y = i(\rho_{ab} - \rho_{ba}) = Tr(\rho\sigma_y)$$

$$z = \rho_{aa} - \rho_{bb} = Tr(\rho\sigma_z)$$

$$x = \sin\theta\cos\phi$$

$$y = \sin\theta\sin\phi$$

$$z = \cos\theta$$

Stokes

vector beam polarimetry

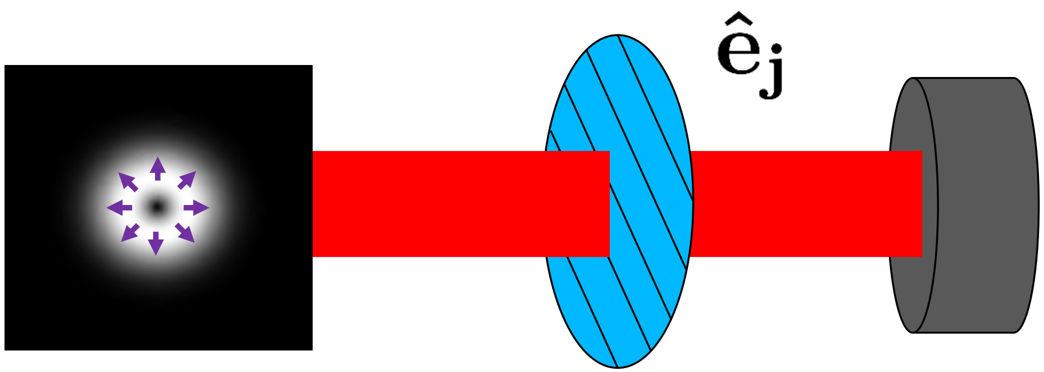
Polarization Stokes parameters for vector beams

$S_1^2 + S_2^2 + S_3^2 = 1$ Fully polarized

$S_1^2 + S_2^2 + S_3^2 < 1$ Partially polarized

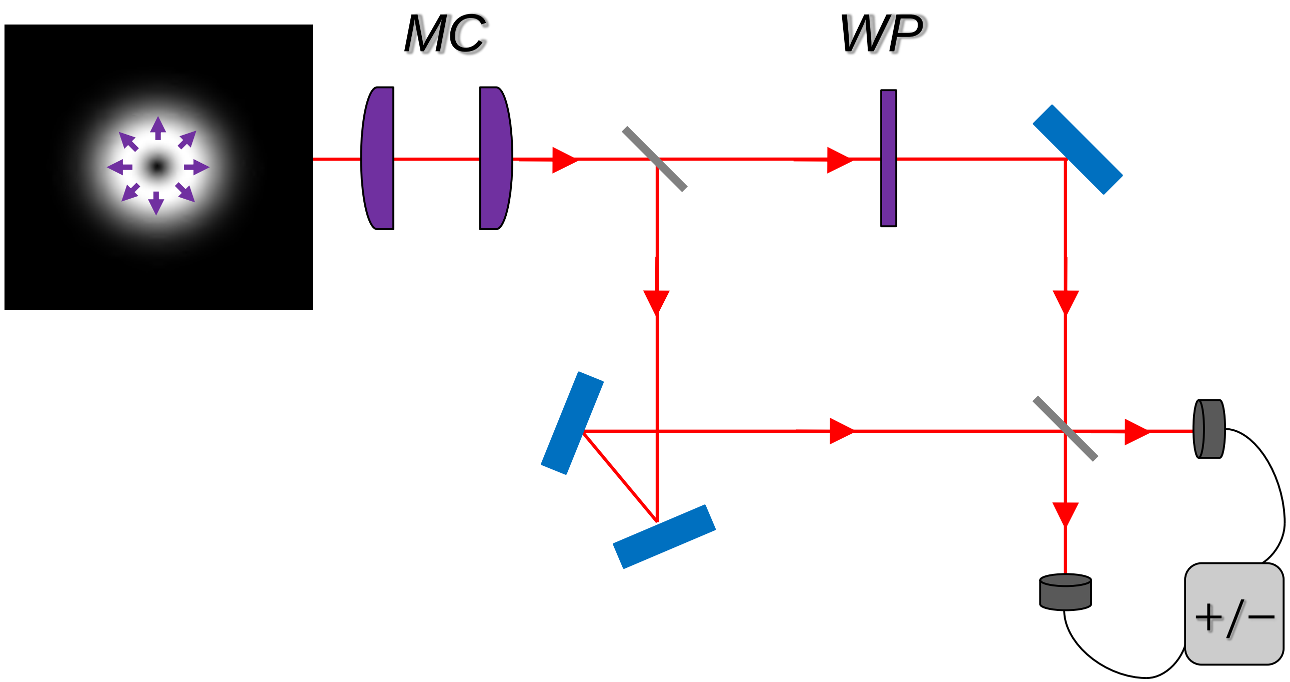
$S_1^2 + S_2^2 + S_3^2 = 0$ Fully unpolarized

Vector beams $S_1^2 + S_2^2 + S_3^2 = ?$



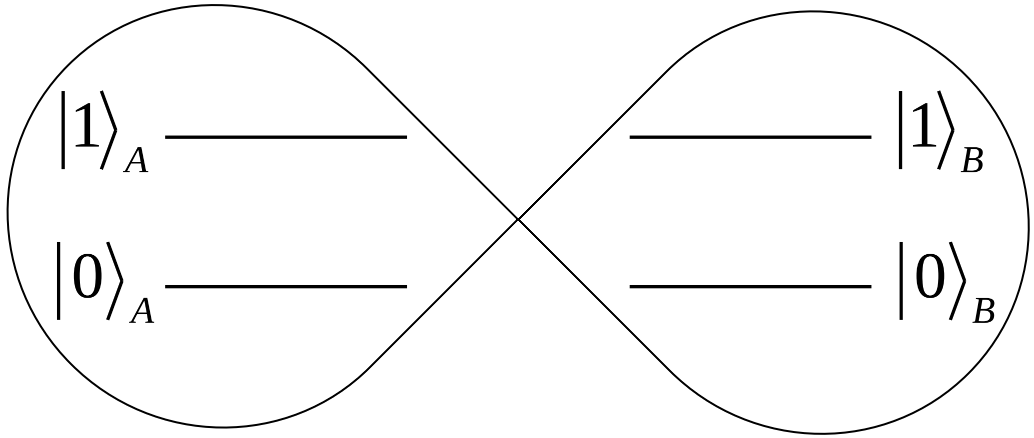
$$I_j = \int \left| \hat{\mathbf{e}}_j^* \cdot \Psi(\mathbf{r}) \right|^2 d^2\mathbf{r} = \sum_k \left| \hat{\mathbf{e}}_j^* \cdot \int \varphi_k^*(\mathbf{r}) \Psi(\mathbf{r}) \, d^2\mathbf{r} \right|^2 \Rightarrow S_1^2 + S_2^2 + S_3^2 = 0 \,!$$

Orbital Stokes parameters for vector beams



$$I_j = \sum_k \left| \hat{\mathbf{e}}_k^* \cdot \int \varphi_j^*(\mathbf{r}) \Psi(\mathbf{r})\, d^2\mathbf{r} \right|^2 \Rightarrow \boxed{O_1^2 + O_2^2 + O_3^2 = 0\,!}$$

Quantum mechanical counterpart



$$\rho_{AB} = \sum_{i,j;k,l} \left(\rho_{AB}\right)_{i,j;k,l} |i\rangle_A |k\rangle_B \langle j|_A \langle l|_B$$

Total D.M.

$$\rho_A = Tr_B\, \rho_{AB} = \sum_k \langle k|_B\, \rho_{AB}\, |k\rangle_B = \sum_{i,j} \sum_k \left(\rho_{AB}\right)_{i,j;k,k} |i\rangle_A \langle j|_A$$

Partial D.M.

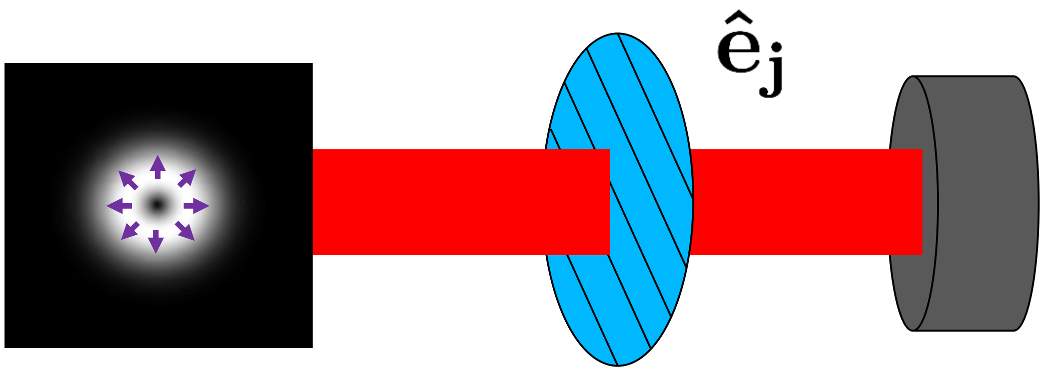
Globally Coherent

$$|\Psi\rangle_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \Rightarrow \rho_A = \rho_B = \frac{\mathbf{1}}{2}$$

Locally Incoherent

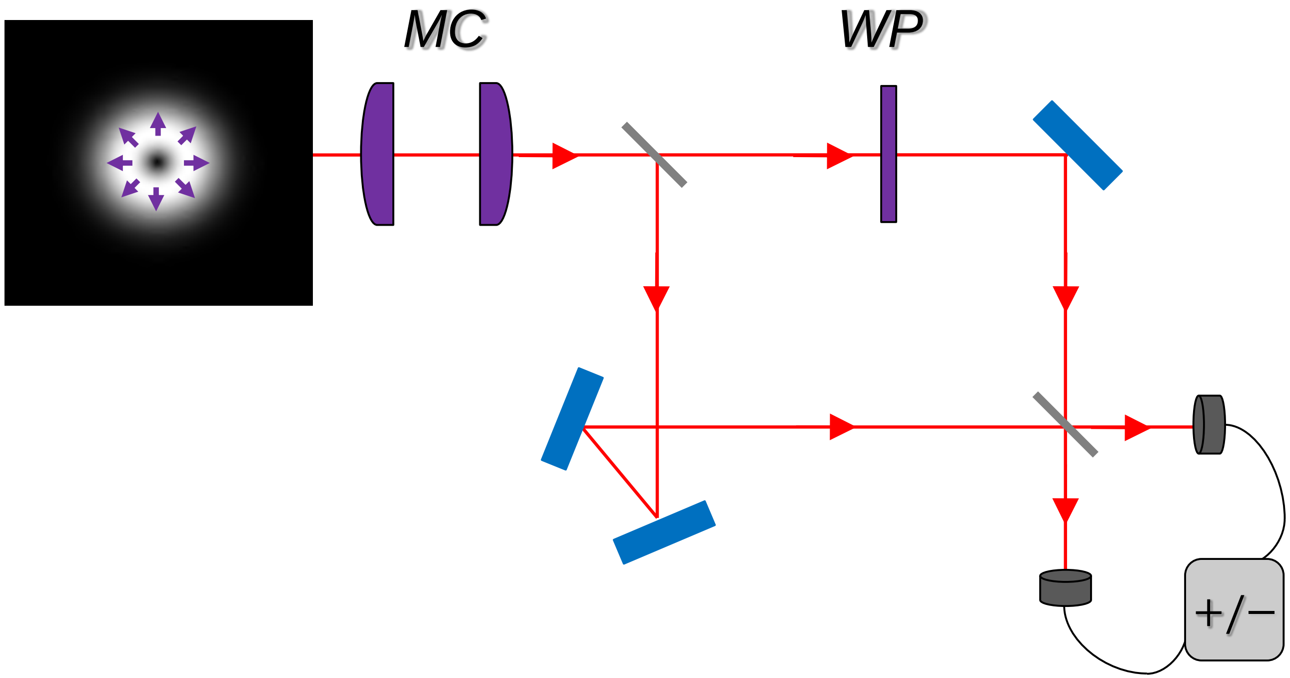
Spatial partial trace

$$I_j = \int \left| \hat{\mathbf{e}}_j^* \cdot \Psi(\mathbf{r}) \right|^2 d^2\mathbf{r} = \underbrace{\sum_k}_{\text{Spatial partial trace}} \left| \hat{\mathbf{e}}_j^* \cdot \int \varphi_k^*(\mathbf{r}) \Psi(\mathbf{r})\, d^2\mathbf{r} \right|^2$$



$$S_1^2 + S_2^2 + S_3^2 = 0\ !$$

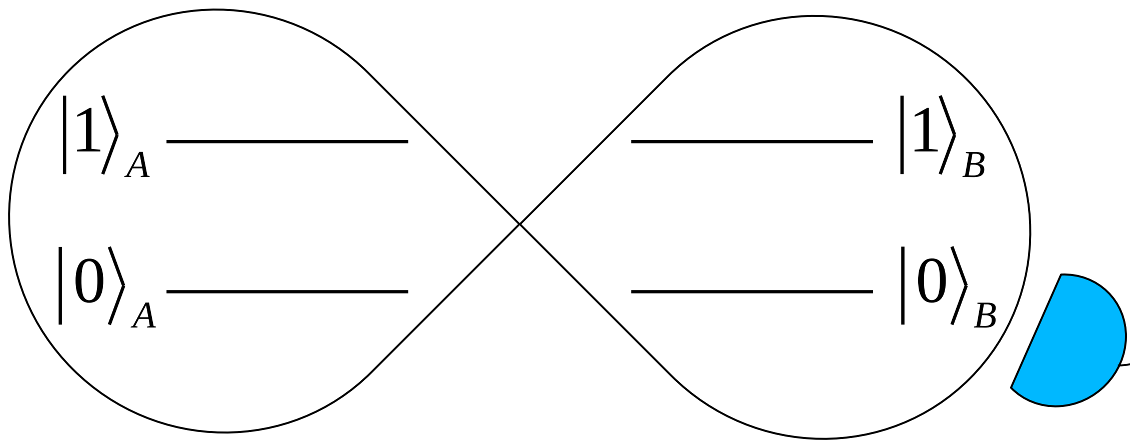
Orbital Stokes parameters for vector beams



$$I_j = \sum_k \left| \hat{\mathbf{e}}_k^* \cdot \int \varphi_j^*(\mathbf{r}) \Psi(\mathbf{r})\, d^2\mathbf{r} \right|^2 \Rightarrow O_1^2 + O_2^2 + O_3^2 = 0\ !$$

Polarization partial trace

Local measurement



Local measurement

$$|\varphi\rangle_B\langle\varphi|_B$$

$$|\Psi\rangle_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \Rightarrow |\psi\rangle_A |\varphi\rangle_B$$

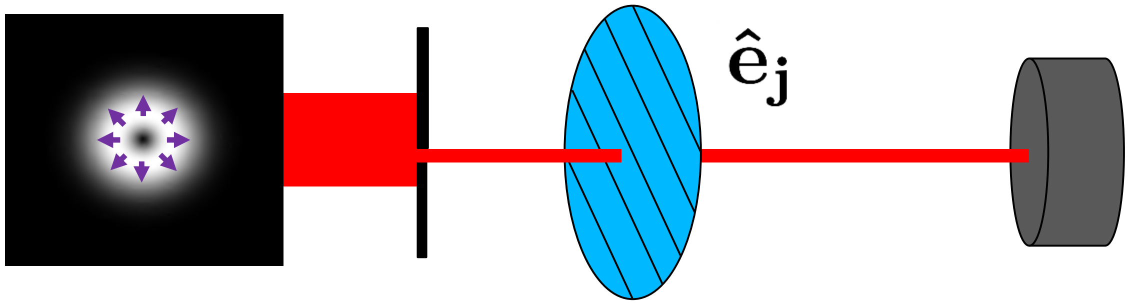
Remote projection

$$|\psi\rangle_A = \langle\varphi|0\rangle_B |0\rangle + \langle\varphi|1\rangle_B |1\rangle$$

General case

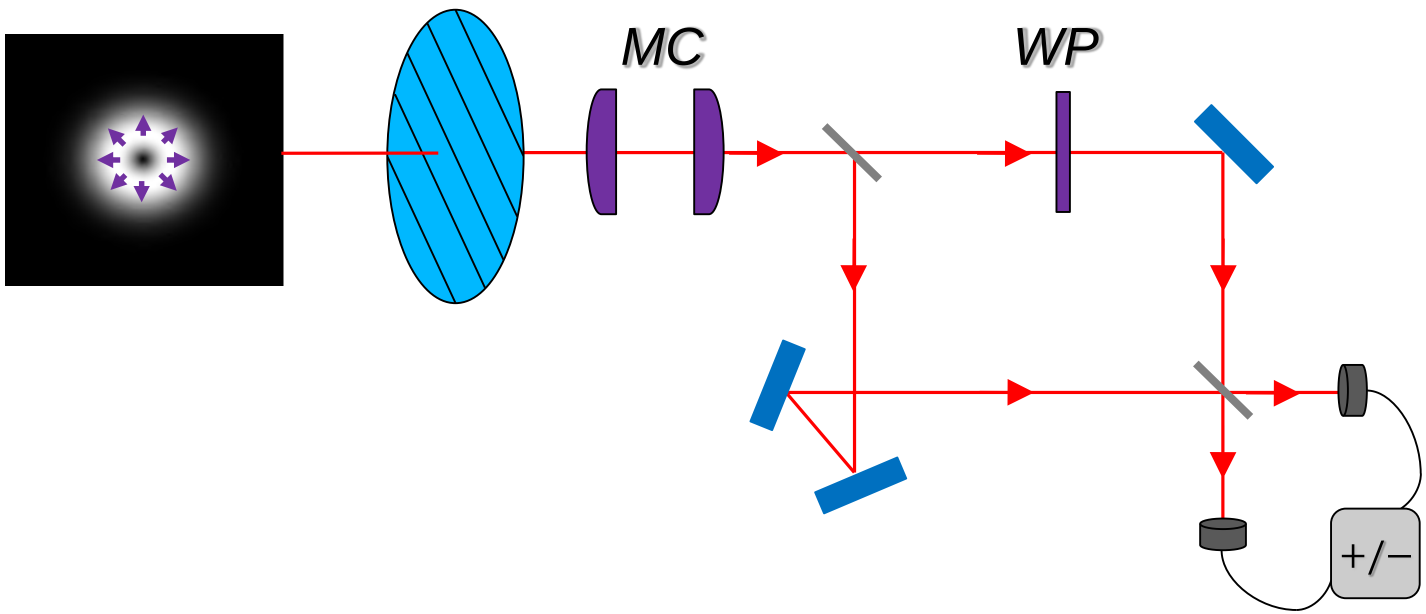
$$\rho_{AB} \rightarrow \left[Tr_B\left(\rho_{AB}|\varphi\rangle\langle\varphi|_B\right)\right]_A \otimes |\varphi\rangle\langle\varphi|_B$$

Spatial and polarization filtering



$$I_j(\mathbf{r}) \Rightarrow S_{1,2,3}(\mathbf{r})$$

$$S_1^2(\boldsymbol{r}) + S_2^2(\boldsymbol{r}) + S_3^2(\boldsymbol{r}) > 0$$



$$O_1^2(\boldsymbol{r}) + O_2^2(\boldsymbol{r}) + O_3^2(\boldsymbol{r}) > 0$$

Rotation invariance

2-qubit invariance

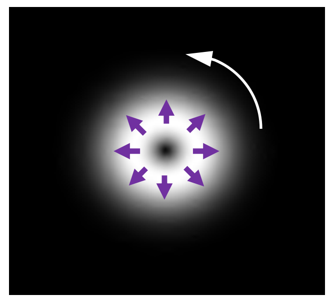
$$\left|\theta\right\rangle = \cos\theta\left|0\right\rangle + \sin\theta\left|1\right\rangle$$
$$\left|\theta'\right\rangle = -\sin\theta\left|0\right\rangle + \cos\theta\left|1\right\rangle$$

$$\left|\Psi_+\right\rangle = \frac{\left|00\right\rangle + \left|11\right\rangle}{\sqrt{2}} = \frac{\left|\theta,\theta\right\rangle + \left|\theta',\theta'\right\rangle}{\sqrt{2}}$$

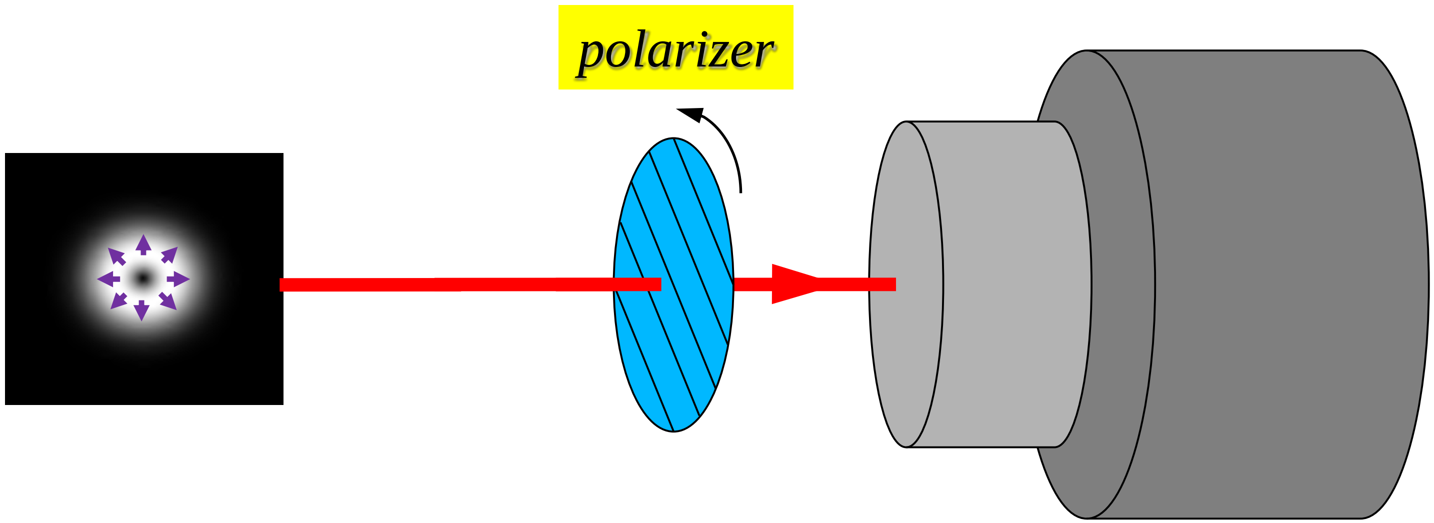
Vector beam invariance

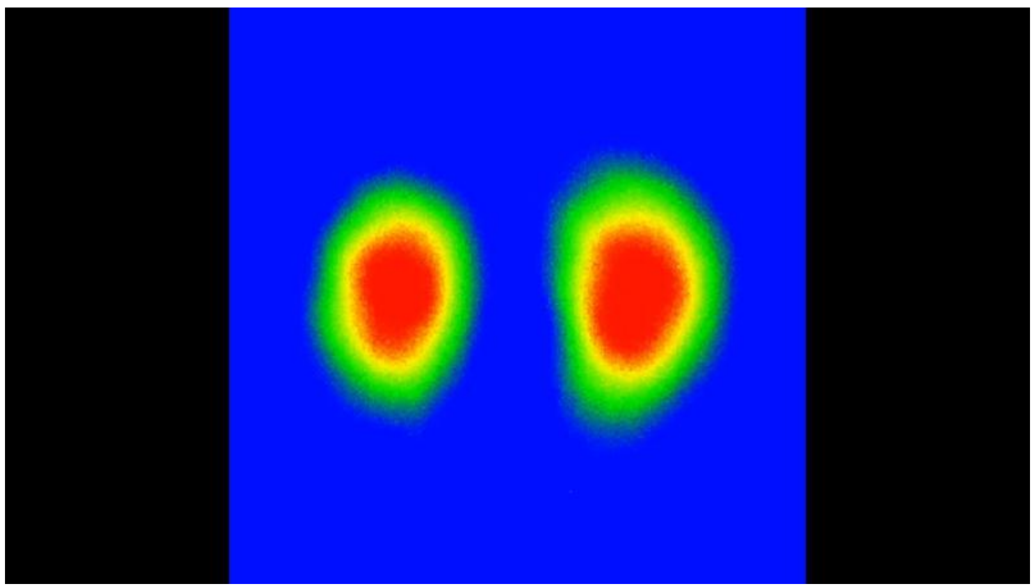
$$\hat{\mathbf{e}}_\theta = \cos\theta\,\hat{\mathbf{e}}_H + \sin\theta\,\hat{\mathbf{e}}_V \qquad \psi_\theta(\mathbf{r}) = \cos\theta\,\psi_H(\mathbf{r}) + \sin\theta\,\psi_V(\mathbf{r})$$
$$\hat{\mathbf{e}}'_\theta = -\sin\theta\,\hat{\mathbf{e}}_H + \cos\theta\,\hat{\mathbf{e}}_V \qquad \psi'_\theta(\mathbf{r}) = -\sin\theta\,\psi_H(\mathbf{r}) + \cos\theta\,\psi_V(\mathbf{r})$$

$$\mathbf{\Psi}_+ = \frac{\psi_H(\mathbf{r})\,\hat{\mathbf{e}}_H + \psi_V(\mathbf{r})\,\hat{\mathbf{e}}_V}{\sqrt{2}} = \frac{\psi_\theta(\mathbf{r})\,\hat{\mathbf{e}}_\theta + \psi'_\theta(\mathbf{r})\,\hat{\mathbf{e}}'_\theta}{\sqrt{2}}$$



Polarization filtering of a vector beam





Creating OAM through polarization operations

2-qubit invariance

$$\left|\theta^{+}\right\rangle = \cos\theta\left|0\right\rangle + i\sin\theta\left|1\right\rangle$$

$$\left|\theta^{-}\right\rangle = -\sin\theta\left|0\right\rangle + i\cos\theta\left|1\right\rangle$$

$$\left|\Psi'_{+}\right\rangle = \frac{\left|00\right\rangle + i\left|11\right\rangle}{\sqrt{2}} = \frac{\left|\theta,\theta^{+}\right\rangle + \left|\theta',\theta^{-}\right\rangle}{\sqrt{2}}$$

Vector beam invariance

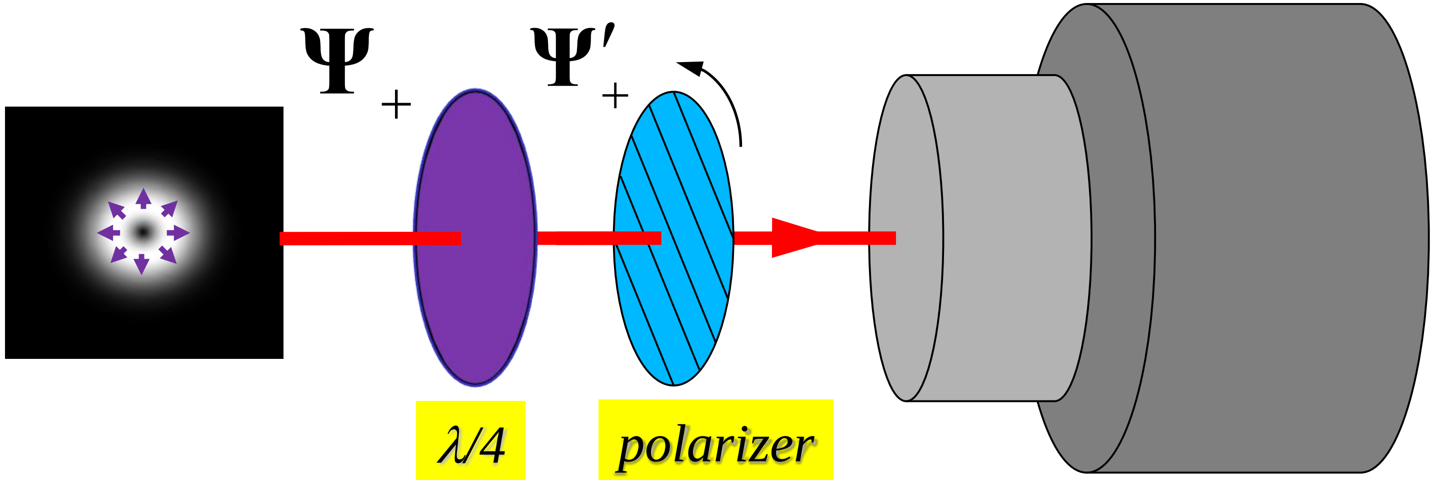
$$\hat{\mathbf{e}}_{\theta} = \cos\theta\,\hat{\mathbf{e}}_{H} + \sin\theta\,\hat{\mathbf{e}}_{V} \qquad \psi_{\theta}^{+}(\mathbf{r}) = \cos\theta\,\psi_{H}(\mathbf{r}) + i\sin\theta\,\psi_{V}(\mathbf{r})$$

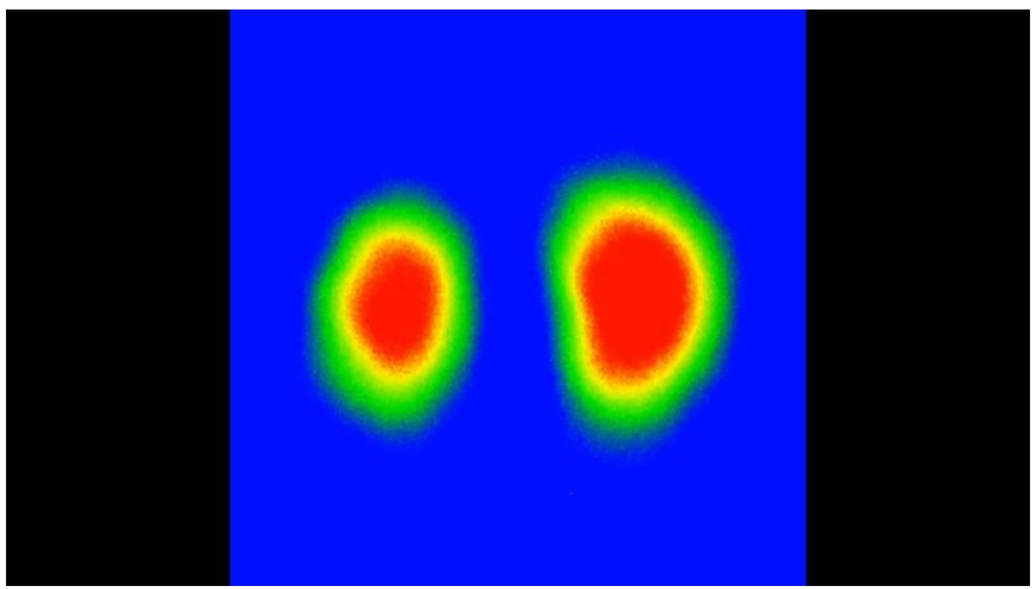
$$\hat{\mathbf{e}}'_{\theta} = -\sin\theta\,\hat{\mathbf{e}}_{H} + \cos\theta\,\hat{\mathbf{e}}_{V} \qquad \psi_{\theta}^{-}(\mathbf{r}) = -\sin\theta\,\psi_{H}(\mathbf{r}) + i\cos\theta\,\psi_{V}(\mathbf{r})$$

$$\mathbf{\Psi}'_{+} = \frac{\psi_{H}(\mathbf{r})\,\hat{\mathbf{e}}_{H} + i\psi_{V}(\mathbf{r})\,\hat{\mathbf{e}}_{V}}{\sqrt{2}} = \frac{\psi_{\theta}^{+}(\mathbf{r})\,\hat{\mathbf{e}}_{\theta} + \psi_{\theta}^{-}(\mathbf{r})\,\hat{\mathbf{e}}'_{\theta}}{\sqrt{2}}$$

$$\theta = 45^{\circ} \Rightarrow \mathrm{OAM}$$

Polarization filtering of a vector beam





Conclusions

- Fundamentals of optical OAM and transverse modes.
- Spin-orbit structural non separability in *classical* laser beams .
- Analogies with principles of quantum measurement theory.